CHAPTER

3

Linear Equations

Overview

In Chapter 3, students use equations to model linear growth and graphs of straight lines and learn the balancing method for solving equations. This chapter builds toward the concept of *function,* which is formalized in Chapter 8.

Lesson 3.1 begins the development of linear growth with the study of recursive sequences. In **Lesson 3.2,** students encounter linear plots. These two ideas are combined through the notion of "walking instructions" to study motion in **Lesson 3.3.** The ideas of starting value and rate of change are formalized in **Lesson 3.4** with the intercept form of a line.

In **Lesson 3.5,** students study rates of change further, using input-output tables that foreshadow the study of functions. **Lesson 3.6** demonstrates the balancing technique for solving equations. In the Activity Day, **Lesson 3.7,** students model real-world data with their linear equations.

The Mathematics

Linearity

Having a *constant rate of change* is a primary characteristic of linearity. You start somewhere and advance by the same amount at each step. This kind of change is represented by a recursive sequence, easily generated on a calculator.

-15 ENTER	Start with -15.
Ans $+10$ ENTER	Add 10 to the answer.
ENTER; ENTER	Continue to add 10 to each answer.

A constant rate of change produces linear growth, though the values will be shrinking instead of growing if the rate of change is negative.

A second way to think about linearity is through equations that relate variables. Students begin to use, write, and make sense of the intercept form of the equation of a line, $y = a + bx$. Seeing and using multiple representations help students connect the recursive sequence start value with a and its constant rate of change with b. The calculator steps for the recursive sequence above are equivalent to the equation $y = -15 + 10x$, when the initial x-value is 0.

The traditional *slope-intercept form,* $y = mx + b$, is mentioned in Lesson 4.2. In Lesson 4.3, students will see the *point-slope form,* $y = y_1 + b(x - x_1)$.

A third way to think about linearity is through *graphs.* Indeed, the term *linearity* comes from the fact that the associated graphs are (straight) lines. Students have seen linear graphs before—in data points, the graph of $y = x$ for comparing estimates with actual distances in Chapter 1, and direct variations $y = kx$ in Chapter 2.

The new forms of equations of a line indicate new ways of thinking of the graph. For example, the intercept form $y = a + bx$ allows students to graph by starting at point $(0, a)$ and moving vertically b units for each unit they move across from left to right. This process reflects the constant rate of change of linear growth and the recursive sequence. Later, in Chapter 8, students will discover that the point-slope equation $y = y_1 + b(x - x_1)$ represents a vertical shift of y_1 and a horizontal shift of x_1.

Most data sets from real-world situations with a linear trend aren't exactly linear. Lesson 3.7 provides an activity for finding an equation that models a set of data points that lie close to but not on a straight line. Students will learn more about lines of fit in Chapter 4.

Solving Equations

Many real-life situations call for predicting when linear growth will reach a certain value. Ways of making that prediction reflect the three ways of thinking about linearity—constant rate of change, equations that relate variables, and graphs.

From the *constant-rate-of-change* perspective, you can run the recursive routine until it reaches the desired output, counting input steps as you go. To mount a flagpole 75 ft up on a building, what floor would you go to if the building's basement floor is 15 ft below the ground and its floors are 10 ft apart?

Just run the sequence -15 ENTER; Ans + 10; ENTER; ENTER; . . . until you get to 75 ft.

To undo those steps and get back to the number of floors, you can subtract -15 and divide by 10, the distance between floors.

This undoing process took place with equations in Chapter 2: If $10x - 15 = 75$, you can "get back to" x by adding 15 to 75 and then dividing by 10. The equation can also be solved using the metaphor of an equation as a pan balance. To keep it balanced, you do the same thing to both sides.

The third approach to linearity, *graphs,* also provides a means of solving equations. You can graph the data points or the equation on a graphing calculator and then use the trace feature to approximate the input value for the desired output value. Using the calculator's table features is another way to approximate a solution.

Using This Chapter

Lesson 3.1 is essential because recursive sequences will be used throughout the book. Solving by undoing is emphasized until the introduction of the balancing method in Lesson 3.6. If you must skip a lesson, you could skip the activity day, Lesson 3.7. No new material is presented, and the rope tying is done again in Lesson 5.2.

Resources

Discovering Algebra Resources

Teaching and Worksheet Masters
Lessons 3.2, 3.5, 3.6

Calculator Notes 0D, 1J, 2A, 2C, 3A, 3B, 3C, 3D

Sketchpad Demonstrations
Lessons 3.1, 3.5, 3.6

Fathom Demonstrations
Lessons 3.1, 3.2, 3.4, 3.7

CBL 2 Demonstration
Lesson 3.5

Dynamic Algebra Explorations online
Lessons 3.1, 3.2

Assessment Resources
Quiz 1 (Lessons 3.1–3.3)
Quiz 2 (Lessons 3.4–3.6)
Chapter 3 Test
Chapter 3 Constructive Assessment Options
Chapters 1 to 3 Exam

More Practice Your Skills for Chapter 3

Condensed Lessons for Chapter 3

Other Resources

Play It Again Sam: Recurrence Equations and Recursion in Mathematics and Computer Science by Rochelle Wilson Meyer and Walter Meyer.

For complete references to this and other resources see www.keypress.com/DA.

Materials

- boxes of toothpicks
- graph paper
- colored pencils
- 4 m measuring tapes, metersticks, or ropes
- motion sensors
- stopwatches or watches with second hands
- 300 pennies or other markers
- washers, *optional*
- paper cups (three per group)
- pieces of rope of different lengths (around 1 m) and thickness (two per group)

Pacing Guide

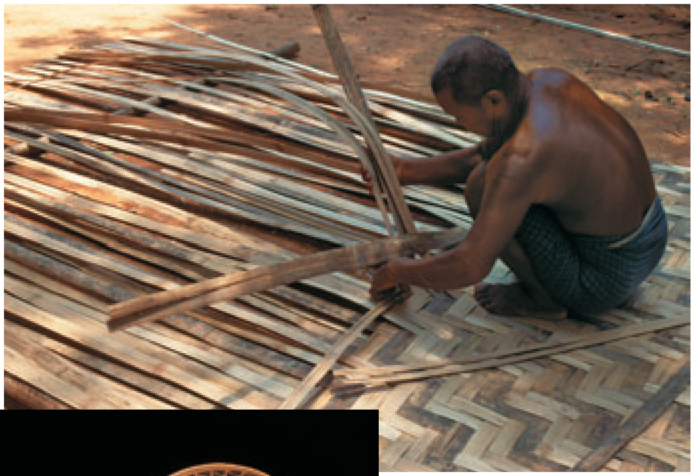
	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9	day 10
standard	3.1	3.2	3.2	3.3	quiz, 3.4	3.5	3.6	3.7	review	assessment
enriched	3.1	3.2	3.2	3.3, project	quiz, 3.4	3.5	3.6	3.7	review, TAL	assessment
block	3.1, 3.2	3.2, 3.3, project	3.4, 3.5	3.6, quiz	3.7, review	assessment, mixed review	exam			

	day 11	day 12
standard	mixed review	exam
enriched	mixed review	exam

CHAPTER

3

Linear Equations



Weavers repeat steps when they make baskets and mats, creating patterns of repeating shapes. This process is not unlike recursion. In the top photo, a mat weaver in Myanmar creates a traditional design with palm fronds. The bottom photo shows bowls crafted by Native American artisans.

OBJECTIVES

In this chapter you will

- write recursive routines emphasizing start plus change
- study rate of change
- learn to write equations for lines using a starting value and a rate of change
- use equations and tables to graph lines
- solve linear equations

CHAPTER 3 OBJECTIVES

- Investigate recursive (arithmetic) sequences using the calculator
- Graph scatter plots of recursive sequences
- Explore time-distance graphs
- Write a linear equation in intercept form given a recursion routine, a graph, or data
- Solve linear equations by undoing operations and by balancing

By keeping the amount of woven material constant, the weavers produce consistent patterns of lines and shapes. To make the mat in the chevron-like pattern, the weaver first lays the warp pieces lengthwise in a repeating pattern of two light and one dark. Then he weaves the woof (or weft) pieces horizontally with the same sequence of two light and one dark. Each woof piece alternately hops over three of the warp pieces and goes under three warp pieces.

Students can try this themselves with colored paper strips and write instructions for how to start the woof piece at the edge.

Coiled baskets begin with a spiral at the base, in contrast to baskets that begin with a wagon-wheel-like pattern at the base. As the basket diameter widens, the space widens between elements that cross the coil. Help students notice that some of the designs, when the baskets are viewed from top or bottom, have reflective (mirror) symmetry, but others have radial symmetry.

Myanmar (Burma) is located between India and Thailand near the Bay of Bengal. The Native American baskets are a Pima coiled tray, an Apache coiled jar, two Yokuts coiled bowls, a Mono coiled bowl, a Maidu coiled bowl, a Washo coiled bowl, and a Pomo coiled gift basket.

LESSON
3.1

Recursive Sequences

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made of ideas.

G. H. HARDY

The Empire State Building in New York City has 102 floors and is 1250 ft high. How high up are you when you reach the 80th floor? You can answer this question using a recursive sequence. In this lesson you will learn how to analyze geometric patterns, complete tables, and find missing values using numerical sequences.

A **recursive sequence** is an ordered list of numbers defined by a starting value and a rule. You generate the sequence by applying the rule to the starting value, then applying it to the resulting value, and repeating this process.

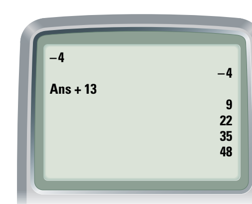
EXAMPLE A

The table shows heights above and below ground at different floor levels in a 25-story building. Write a **recursive routine** that provides the sequence of heights −4, 9, 22, 35, . . . , 217, . . . that corresponds to the building floor numbers 0, 1, 2, ... Use this routine to find each missing value in the table.

Floor number	Basement (0)	1	2	3	4	. . .	10	. . .		. . .	25
Height (ft)	−4	9	22	35		. . .		. . .	217	. . .	

▶ Solution

The starting value is −4 because the basement is 4 ft below ground level. Each floor is 13 ft higher than the floor below it, so the rule for finding the next floor height is "add 13 to the current floor height."



The calculator screen shows how to enter this recursive routine into your calculator. Press −4 ENTER to start your number sequence. Press +13 ENTER. The calculator automatically displays Ans +13 and computes the next value. Simply pressing ENTER again applies the rule for finding successive floor heights. [▶ See Calculator Note 0D. ◀] You can see that the 4th floor is at 48 ft.

How high up is the 10th floor? Count the number of times you press ENTER until you reach 10. Which floor is at a height of 217 ft? Keep counting until you see that value on your calculator screen. What's the height of the 25th floor? Keep applying the rule by pressing ENTER and record the values in your table.

The 10th floor is at 126 ft, the 17th floor is at 217 ft, and the 25th floor is at 321 ft.

LESSON
3.1

PLANNING

LESSON OUTLINE

One day:

5 min	Introduction, Example A
20 min	Investigation
10 min	Sharing, Example B
5 min	Closing
10 min	Exercises

MATERIALS

- boxes of toothpicks (or have students draw line segments on paper)
- Calculator Note 0D
- Sketchpad demonstration Patterns and Recursion, *optional*
- Fathom demonstration Recursive Sequences, *optional*

TEACHING

In this lesson the idea of recursion, introduced in Chapter 0 with fractals and evaluation of expressions, is revisited with calculator home-screen iteration of recursive sequences.

INTRODUCTION

Students may say that they need only a proportion to determine the height of the 80th floor of the Empire State Building. However, they would be making some assumptions. Leave that question open until Exercise 5.

If you skipped Chapter 0, explain that while executing a *recursive procedure* you do the same thing repeatedly, at each step operating on the result of the previous step.

▶ *EXAMPLE A*

[ELL] *Floor,* or *story,* means the level of a building, usually beginning with 1. The *height of a floor* could mean the floor-to-ceiling distance of one floor, but here it

LESSON OBJECTIVES

- Review or become familiar with the concept of recursion
- Investigate recursive (arithmetic) sequences using the calculator

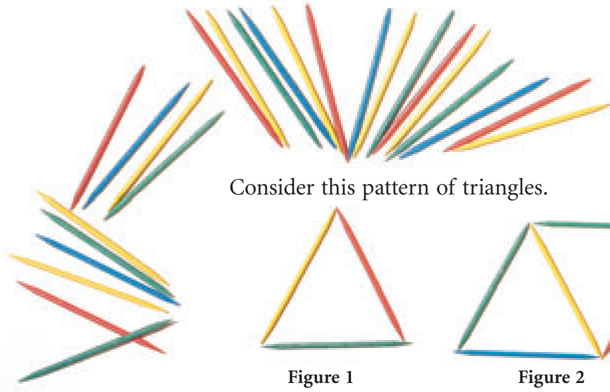
NCTM STANDARDS

	CONTENT		PROCESS
✓	Number		Problem Solving
✓	Algebra	✓	Reasoning
✓	Geometry		Communication
	Measurement		Connections
	Data/Probability	✓	Representation

Investigation
Recursive Toothpick Patterns

You will need
- a box of toothpicks

In this investigation you will learn to create and apply recursive sequences by modeling them with puzzle pieces made from toothpicks.



Consider this pattern of triangles.

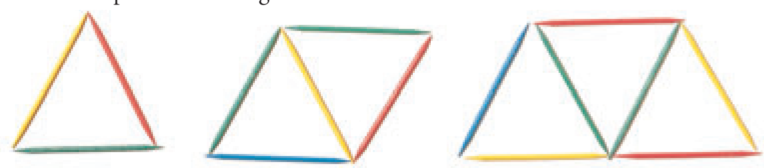


Figure 1 Figure 2 Figure 3

keymath.com/DA

Step 1 Make Figures 1–3 of the pattern using as few toothpicks as possible. How many toothpicks does it take to reproduce each figure? How many toothpicks lie on the perimeter of each figure?

Step 2 Copy the table with enough rows for six figures of the pattern. Make Figures 4–6 from toothpicks by adding triangles in a row and complete the table.

	Number of toothpicks	Perimeter
Figure 1		
Figure 2		

Step 3 What is the rule for finding the number of toothpicks in each figure? What is the rule for finding the perimeter? Use your calculator to create recursive routines for these rules. Check that these routines generate the numbers in your table.

Step 3 Toothpicks: Add 2 to the previous number. Perimeter: Add 1 to the previous perimeter. Press 3 ENTER and then Ans + 2 ENTER to find the successive numbers of toothpicks. For the perimeter, press 3 ENTER and then Ans + 1 ENTER.

Step 4 Now make Figure 10 from toothpicks. Count the number of toothpicks and find the perimeter. Does your calculator routine give the same answers? Find the number of toothpicks and the perimeter for Figure 25. Figure 10: 21 toothpicks with a perimeter of 12; Figure 25: 51 toothpicks with a perimeter of 27

Next you'll see what sequences you can generate with a new pattern.

Step 5 Design a pattern using a row of squares, instead of triangles, with your toothpicks. Repeat Steps 1–4 and answer all the questions with the new design.

Step 6 Choose a unit of measurement and explain how to calculate the area of a square made from toothpicks. How does your choice of unit affect calculations for the areas of each figure?

Step 3 As needed, ask what the starting values for the number of toothpicks and for the perimeter are. Then ask about the rules for finding the next numbers.

Step 5 Be sure the pattern of squares remains a row.

Step 2

Toothpicks	Perimeter
3	3
5	4
7	5
9	6
11	7
13	8

Step 5

Toothpicks	Perimeter
4	4
7	6
10	8
13	10
16	12
19	14

means the *height above ground level* of that floor. This example reminds students of how to use home-screen iteration. You may choose to do it as a class, especially if you skipped recursion in Chapter 0.

Students might wonder if −4 is the first term of the sequence. Point out that sequences can have "zeroth" terms to make the counting easier.

Guiding the Investigation

If you think your students will not handle toothpicks appropriately, you can have them draw line segments on paper.

One Step
Draw the three figures showing the growth of a triangle pattern or ask students to look at page 159. Point out how the perimeter is changing. Ask students to design their own toothpick pattern with a changing feature, such as perimeter, and to write a recursive routine to make their calculator generate the sequence of numbers. As you interact, be sure students see the idea of a common difference between consecutive terms.

Step 1 If students do not use just one toothpick per side, their answers will be multiples of 3.

Step 2 Be sure students extend their patterns of shapes horizontally. If they add shapes in several directions, their sequences will not be arithmetic.

The table is organized so that the calculated sequences are displayed vertically in columns. Calculators will also display the results of recursive routines vertically on the home screen.

Step 5 Toothpicks: Add 3 to the previous number. Perimeter: Add 2 to the previous perimeter. Press 4 ENTER and then Ans + 3 ENTER to find the number of toothpicks. To find the perimeter, press 4 ENTER and then Ans + 2 ENTER. Figure 10: 31 toothpicks with 22 on the perimeter; Figure 25: 76 toothpicks with 52 on the perimeter.

Step 6 Students might measure a toothpick in centimeters, in inches, or as a unit 1 toothpick long. For a unit of 1 toothpick, the area is equal to the number of squares in the figure. Otherwise, the number of squares must be multiplied by the unit area of each square to calculate the area of the entire figure.

Step 7 Students might be reluctant to use figures whose areas are difficult to find in terms of the edges. Suggest that they can use the area of the first figure as 1 square unit even if their basic shape is not a square.

SHARING IDEAS

Ask several students to present their questions from Step 8. For one of them in which the perimeter increases by a different number from the number of toothpicks, ask why. Usually one or more toothpicks in the perimeter at the previous step are no longer in the perimeter.

You might begin to use the term *rate of change* to describe what's happening—for example, the amount being added to the perimeter is the rate of change of the perimeter, in toothpicks per stage.

If you ask some students to put up their sequences with terms missing and have the class guess the missing terms, you can anticipate Example B.

Now you'll create your own puzzle piece from toothpicks. Add identical pieces in one direction to make the succeeding figures of your design.

Step 7 Draw Figures 1–3 on your paper. Write recursive routines to generate number sequences for the number of toothpicks, perimeter, and area of each of six figures. Record these numbers in a table. Find the values for a figure made of ten puzzle pieces.

Step 8 Write three questions about your pattern that require recursive sequences to answer. For example: What is the perimeter if the area is 20? Test your questions on your classmates.

In the investigation you wrote number sequences in table columns. Remember that you can also display sequences as a list of numbers like this:

1, 3, 5, 7, . . .

Each number in the sequence is called a **term.** The three periods indicate that the numbers continue.

EXAMPLE B

Find the missing values in each sequence.

a. 7, 12, 17, __ , 27, __ , __ , 42, __ , 52

b. 5, 1, −3, __ , −11, −15, __ , __ , −27, __

c. −7, __ , −29, __ , −51, −62, __ , −84, __

d. 2, −4, 8, −16, 32, __ , 128, −256, __ , __

How many hidden numbers can you find?

▶ *Solution*

For each sequence, identify the starting value and the operation that must be performed to get the next term.

a. The starting value is 7 and you add 5 each time to get the next number. The missing numbers are shown in red.

starting value

+ 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5

7, 12, 17, 22, 27, 32, 37, 42, 47, 52

b. The starting value is 5 and you subtract 4 each time to get the next number. The missing numbers are shown in red.

starting value

− 4 − 4 − 4 − 4 − 4 − 4 − 4 − 4 − 4

5, 1, −3, −7, −11, −15, −19, −23, −27, −31

Assessing Progress

Watch for systematic data collection, familiarity with geometric shapes and terms such as *perimeter* and *area,* and ability to contribute to group work.

▶ *EXAMPLE B*

This example is good for students who may not have grasped the nature of a sequence with common differences of consecutive terms. Many students enjoy creating and solving problems like this. Encourage differing approaches to part c. You can find the common difference by looking down the list to the first consecutive pair or by finding half the difference between the third term and the first term: $\frac{1}{2}[-29 - (-7)] = -11$. Hidden numbers in the illustration include 0, 1, 2, 3, 4, 7, 8, and 11.

c. The starting value is -7. The difference between the fifth and sixth terms shows that you subtract 11 each time.

starting value

$$-7 \xrightarrow{-11} -18 \xrightarrow{-11} -29 \xrightarrow{-11} -40 \xrightarrow{-11} -51 \xrightarrow{-11} -62 \xrightarrow{-11} -73 \xrightarrow{-11} -84 \xrightarrow{-11} -95$$

-7, -18, -29, -40, -51, -62, -73, -84, -95

d. Adding or subtracting numbers does not generate this sequence. Notice that the numbers double each time. Also, they switch between positive and negative signs. So the rule is to multiply by -2. Multiply 32 by -2 to get the first missing value of -64. The last missing values are 512 and -1024.

starting value

$$2 \xrightarrow{\cdot(-2)} -4 \xrightarrow{\cdot(-2)} 8 \xrightarrow{\cdot(-2)} -16 \xrightarrow{\cdot(-2)} 32 \xrightarrow{\cdot(-2)} -64 \xrightarrow{\cdot(-2)} 128 \xrightarrow{\cdot(-2)} -256 \xrightarrow{\cdot(-2)} 512 \xrightarrow{\cdot(-2)} -1024$$

2, -4, 8, -16, 32, -64, 128, -256, 512, -1024

EXERCISES

You will need your graphing calculator for Exercises **2, 5,** and **7.**

Practice Your Skills

1. Evaluate each expression without using your calculator. Then check your result with your calculator.

a. $-2(5-9)+7$ 15 **b.** $\dfrac{(-4)(-8)}{-5+3}$ -16 **c.** $\dfrac{5+(-6)(-5)}{-7}$ -5

2. Consider the sequence of figures made from a row of pentagons.

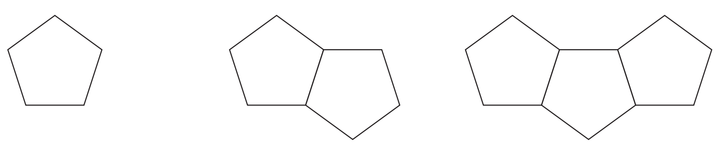


Figure 1 Figure 2 Figure 3

a. Copy and complete the table for five figures. @

b. Write a recursive routine to find the perimeter of each figure. Assume each side is 1 unit long.

c. Find the perimeter of Figure 10. @ 32

d. Which figure has a perimeter of 47? Figure 15

Figure number	Perimeter
1	5
2	8
3	11
4	14
5	17

3. Find the first six values generated by the recursive routine

-14.2 ENTER

Ans + 3.7 ENTER, ENTER, . . . @ $-14.2, -10.5, -6.8, -3.1, 0.6, 4.3$

4. Write a recursive routine to generate each sequence. Then use your routine to find the 10th term of the sequence.

a. 3, 9, 15, 21, . . . @ **b.** 1.7, 1.2, 0.7, 0.2, . . . @

c. -3, 6, -12, 24, . . . **d.** 384, 192, 96, 48, . . .

Closing the Lesson

You generate a **recursive sequence** on the calculator by entering a starting number and then designating how to operate on one number to get the next number in the sequence.

BUILDING UNDERSTANDING

Students practice writing recursive routines to generate sequences, most of which have a constant difference between consecutive terms.

ASSIGNING HOMEWORK

Essential	1–4, 6
Performance assessment	5, 7, 8
Portfolio	6
Journal	10, 11
Group	6, 9, 11, 12
Review	13, 14

Helping with the Exercises

For more practice with recursive geometric sequences, you can use the Sketchpad demonstration Patterns and Recursion.

2b. 5 ENTER, Ans + 3 ENTER, ENTER, . . .

Exercise 4 In 4d, students see a recursive sequence defined by division for the first time. The Fathom demonstration Recursive Sequences can be used to replace this exercise.

4a. Start with 3, then apply the rule Ans + 6; 10th term = 57.

4b. Start with 1.7, then apply the rule Ans − 0.5; 10th term = −2.8.

4c. Start with −3, then apply the rule Ans · −2; 10th term = 1536.

4d. Start with 384, then apply the rule Ans/2 or Ans · 0.5; 10th term = 0.75.

Exercise 5 Students may have difficulty calculating the heights of the floors. Encourage them to draw a picture to see that there are 16 floors, numbered 86 through 101, spanning a distance of 174 ft, and that the lower 85 floors cover a distance of 1050 ft. If students notice the height here of 1224 and the height in the introduction of 1250, say that the height 1224 is to the floor of the 102nd floor and 1250 includes the height of the antenna.

5a. The recursive routine is 0 ENTER and then Ans + 12.35 ENTER. The starting value is 0, the height of ground level (the first floor). Add the average floor height for the next 85 floors: 12.35 ft.

5b. The recursive routine is 1050 ENTER and then Ans + 10.875 ENTER. The starting value is the height of the 86th floor. Add 10.875, the average floor height of floors 86 through 101.

5c. When you are 531 ft high, you are 43 floors up and thus on the 44th floor.

6a. Possible explanation: The smallest square has an area of 1. The next larger white square has an area of 4, which is 3 more than the smallest square. The next larger gray square has an area of 9, which is 5 more than the 4-unit white square.

6b. The recursive routine is 1 ENTER, Ans + 2 ENTER, ENTER, and so on.

6c. 17, the value of the 9th term in the sequence

Reason and Apply

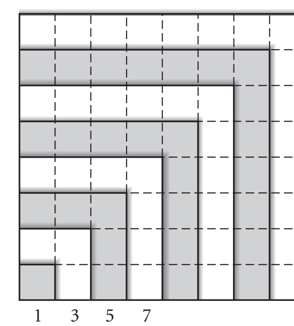
5. APPLICATION In the Empire State Building the longest elevator shaft reaches the 86th floor, 1050 ft above ground level. Another elevator takes visitors from the 86th floor to the observation area on the 102nd floor, 1224 ft above ground level. For more information about the Empire State Building, see www.keymath.com/DA.

a. Write a recursive routine that gives the height above ground level for each of the first 86 floors. Tell what the starting value and the rule mean in terms of the building.

b. Write a recursive routine that gives the heights of floors 86 through 102. Tell what the starting value and the rule mean in this routine.

c. When you are 531 ft above ground level, what floor are you on?

d. When you are on the 90th floor, how high up are you? When you are 1137 ft above ground level, what floor are you on?
1093.5 ft; 94th floor



6. The diagram at right shows a sequence of gray and white squares each layered under the previous one.

a. Explain how the sequence 1, 3, 5, 7, . . . is related to the areas of these squares. @

b. Write a recursive routine that gives the sequence 1, 3, 5, 7, ... @

c. Use your routine to predict the number of additional unit squares you would need to enlarge this diagram by one additional row and column. Explain how you found your answers. @

d. What is the 20th number in the sequence 1, 3, 5, 7, . . . ? 39

e. The first term in the sequence is 1, and the second is 3. Which term is the number 95? Explain how you found your answer.
The 48th term is 95; students might press ENTER 48 times or compute 2(48) − 1.

7. Imagine a tilted L-shaped puzzle piece made from 8 toothpicks. Its area is 3 square units. Add puzzle pieces in the corner of each "L" to form successive figures of the design. In a second figure, the two pieces "share" two toothpicks so that there are 14 toothpicks instead of 16.

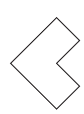


Figure 1

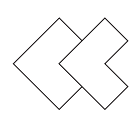


Figure 2

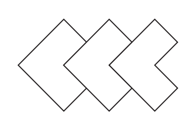


Figure 3

a. As you did in the investigation, make a table with enough columns and rows for the number of toothpicks, perimeter, and area of each of six figures.

b. Write a recursive routine that will produce the number sequence in each column of the table.

c. Find the number of toothpicks, perimeter, and area of Figure 10.

d. Find the perimeter and area of the figure made from 152 toothpicks.

7a. The table for six figures of the L-shaped puzzle pieces is

Figure	Toothpicks	Perimeter	Area
1	8	8	3
2	14	12	6
3	20	16	9
4	26	20	12
5	32	24	15
6	38	28	18

7b. To find the number of toothpicks, press 8 ENTER and then Ans + 6 ENTER. To find the perimeter, press 8 ENTER and then Ans + 4 ENTER. For the area, press 3 ENTER and then Ans + 3 ENTER.

7c. Figure 10 has 62 toothpicks, a perimeter of 44, and an area of 30.

7d. Figure 25, made from 152 toothpicks, has a perimeter of 104 and an area of 75.

8. **APPLICATION** The table gives some floor heights in a building.

Floor	. . .	-1	0	1	2	. . .		. . .	25
Height (m)	. . .	-3	1	5	9	. . .	37	. . .	

a. How many meters are between the floors in this building? 4 m

b. Write a recursive routine that will give the sequence of floor heights if you start at the 25th floor and go to the basement (floor 0). Which term in your sequence represents the height of the 7th floor? What is the height?

c. How many terms are in the sequence in 8b? 26 terms

d. Floor "-1" corresponds to the first level of the parking substructure under the building. If there are five parking levels, how far underground is level 5? 19 m

9. Consider the sequence __, -4, 8, __, 32, ...

a. Find two different recursive routines that could generate these numbers. ⓗ

b. For each routine, what are the missing numbers? What are the next two numbers?

c. If you want to generate this number sequence with exactly one routine, what more do you need? More numbers are needed to uniquely determine a recursive routine.

10. Positive multiples of 7 are generally listed as 7, 14, 21, 28, ...

a. If 7 is the 1st multiple of 7 and 14 is the 2nd multiple, then what is the 17th multiple? @ $17 \cdot 7$, or 119

b. How many multiples of 7 are between 100 and 200? @ 14

c. Compare the number of multiples of 7 between 100 and 200 with the number between 200 and 300. Does the answer make sense? Do all intervals of 100 have this many multiples of 7? Explain. @

d. Describe two different ways to generate a list containing multiples of 7. @

11. Some babies gain an average of 1.5 lb per month during the first 6 months after birth.

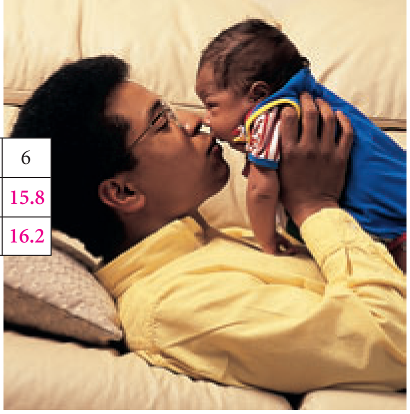
a. Write a recursive routine that will generate a table of monthly weights for a baby weighing 6.8 lb at birth. Press 6.8 ENTER and then Ans + 1.5 ENTER, ENTER . . . .

b. Write a recursive routine that will generate a table of monthly weights for a baby weighing 7.2 lb at birth. Press 7.2 ENTER, and then Ans + 1.5 ENTER, ENTER . . . .

c. How are the routines in 11a and 11b the same? How are they different? The starting terms differ; the rule itself is the same.

d. Copy and complete the table of data for this situation.

Age (mo)	0	1	2	3	4	5	6
Weight of Baby A (lb)	6.8	8.3	9.8	11.3	12.8	14.3	15.8
Weight of Baby B (lb)	7.2	8.7	10.2	11.7	13.2	14.7	16.2



e. How are the table values for the two babies the same? How do they differ?
Each baby always increases by 1.5 lb, and the difference between the babies' weights is always 0.4 lb; the starting values are different.

Exercise 8b In other words, let the 25th floor height be the first term, the 24th floor height be the second term, and so on.

8b. Press 101 ENTER and then Ans − 4 ENTER. The 19th term represents the height of the 7th floor. The height is 29 m.

9a. One routine is press -16 ENTER and then Ans + 12 ENTER. Another is press 2 ENTER and then Ans $\cdot$ -2 ENTER.

9b. Two possible sequences are $\{-16, -4, 8, 20, 32, 44, 56, \ldots\}$ and $\{2, -4, 8, -16, 32, -64, 128, \ldots\}$.

Exercise 10 If appropriate, ask about multiples of 7 that aren't positive. Mention nonnegative multiples (which include 0) as well as negative multiples.

10c. Possible answer: There are 14 multiples between 100 and 200. There are also 14 multiples of 7 between 200 and 300, but there are 15 between 300 and 400.

10d. Possible answer: The 4th multiple of 7 is $4 \cdot 7$, or 28; the 5th multiple of 7 is $5 \cdot 7$, or 35; and so on. Recursively, you start with 7 and then continue adding 7.

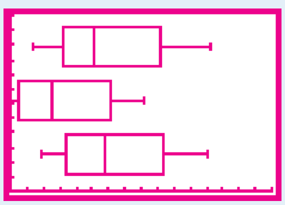
12a. Press 1 (ENTER), Ans · 3 (ENTER), (ENTER) . . . ; the 9th term is 6561.

12b. Press 5 (ENTER), Ans · (−1) (ENTER), (ENTER) . . . ; the 123rd term is 5.

12c. Press −16.2 (ENTER), Ans + 1.4 (ENTER), (ENTER). . . ; the 13th term is the first positive term.

Exercise 13 **[ELL]** *Precipitation* is rain or snow.

13a. The top box plot is Portland, the middle is San Francisco, and the bottom is Seattle.



[0, 7, 0.5, 0, 12, 1]

San Francisco has the least precipitation and is the only city in which there is a month with no precipitation. One indicator that the weather is much drier in San Francisco is that the month with no precipitation is not an outlier.

13b. You lose information about what time of year is wettest; a bar graph or scatter plot would show trends over the months of the year more clearly.

12. Write recursive routines to help you answer 12a–d.

a. Find the 9th term of 1, 3, 9, 27, ... @

b. Find the 123rd term of 5, −5, 5, −5, ... @

c. Find the term number of the first positive term of the sequence −16.2, −14.8, −13.4, −12, ...

d. Which term is the first to be either greater than 100 or less than −100 in the sequence −1, 2, −4, 8, −16, . . . ? Press −1 (ENTER), Ans · (−2) (ENTER), (ENTER) . . . ; the 8th term, 128, is the first to be greater than 100.

Review

1.3 **13.** The table gives the normal monthly precipitation for three cities in the United States.

a. Display the data in three box plots, one for each city, and use them to compare the precipitation for the three cities.

b. What information do you lose by displaying the data in a box plot? What type of graph might be more helpful for displaying the data?

Precipitation for Three Cities

	Precipitation (in.)		
Month	**Portland, Oregon**	**San Francisco, California**	**Seattle, Washington**
January	5.4	4.1	5.4
February	3.9	3.0	4.0
March	3.6	3.1	3.8
April	2.4	1.3	2.5
May	2.1	0.3	1.8
June	1.5	0.2	1.6
July	0.7	0.0	0.9
August	1.1	0.1	1.2
September	1.8	0.3	1.9
October	2.7	1.3	3.3
November	5.3	3.2	5.7
December	6.1	3.1	6.0

(*The New York Times Almanac 2000*, pp. 480–481)

It's a rainy day in Portland, Oregon.

2.8 **14.** Create an undo table and solve the equation listed by undoing the order of operations. $x = -2.6$

Equation: $8 + 3(x - 5) = -14.8$		
Description	Undo	Result
Pick x.		−2.6
− (5)	+ (5)	−7.6
· (3)	/ (3)	−22.8
+ (8)	− (8)	−14.8

LESSON

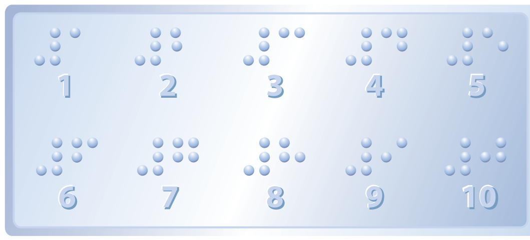
3.2

Linear Plots

In most sciences, one generation tears down what another has built, and what one has established, the next undoes. In mathematics alone, each generation builds a new story to the old structure.

HERMANN HANKEL

In this lesson you will learn that the starting value and the rule of a recursive sequence take on special meaning in certain real-world situations. When you add or subtract the same number each time in a recursive routine, consecutive terms change by a constant amount. Using your calculator, you will see how the starting value and rule let you generate data for tables quickly. You will also plot these data sets and learn that the starting value and rule relate to characteristics of the graph.



Many elevators use Braille symbols. This alphabet for the blind was developed by Louis Braille (1809–1852). For more information about Braille, see the links at www.keymath.com/DA .

EXAMPLE

You walk into an elevator in the basement of a building. Its control panel displays "0" for the floor number. As you go up, the numbers increase one by one on the display, and the elevator rises 13 ft for each floor. The table shows the floor numbers and their heights above ground level.

Floor number	Height (ft)
0 (basement)	−4
1	9
2	22
3	35
4	48
. . .	. . .

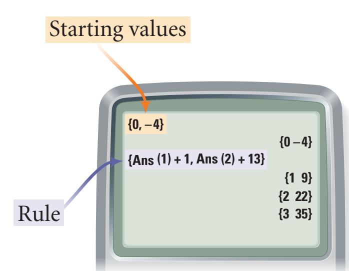
a. Write recursive routines for the two number sequences in the table. Enter both routines into calculator lists.

b. Define variables and plot the data in the table for the first few floors of the building. Does it make sense to connect the points on the graph?

c. What is the highest floor with a height less than 200 ft? Is there a floor that is exactly 200 ft high?

▶ Solution

The starting value for the floor numbers is 0, and the rule is to add 1. The starting value for the height is −4, and the rule is to add 13. You can generate both number sequences on the calculator using lists.



a. Press {0, −4} and press ENTER to input both starting values at the same time. To use the rules to get the next term in the sequence, press {Ans(1) + 1, Ans(2) + 13} ENTER.
[▶ See **Calculator Note 3A.** ◀]

These commands tell the calculator to add 1 to the first term in the list and to add 13 to the second number. Press ENTER again to compute the next floor number and its corresponding height as the elevator rises.

In this routine, the calculator displays a new list of numbers horizontally every time you press ENTER, but the terms for the sequences of floor numbers and heights appear vertically aligned on the screen.

Be sure students understand the important difference between the use of braces and parentheses on the calculator.

Emphasize the use of dimensions: $\frac{height}{floor}$. On the Elevator Table transparency, you might make marks between consecutive terms in the right column to show the common differences.

Ask why the term *linear relationship* was chosen for two quantities in which each unit increase in one results in a constant increase in the other. An answer to this question could wait until Sharing.

LESSON

3.2

PLANNING

LESSON OUTLINE

First day:

5 min Introduction, Example

45 min Investigation

Second day:

15 min Investigation

20 min Sharing, Closing

15 min Exercises

MATERIALS

- graph paper
- colored pencils
- Elevator Table (T), *optional*
- On the Road Again Grid (W)
- On the Road Again Table (W)
- On the Road Again Graph (T)
- Calculator Notes 0D, 2C, 3A
- Fathom demonstration On the Road Again, *optional*

TEACHING

This lesson describes quantities generated by a recursive additive sequence (that is, with a constant rate of change) as having a *linear relationship* because their graph is a set of points that lie on a straight line. If the amount being added (the rate of change per step) is positive, the line is rising (increasing) from left to right, with larger rates of change giving steeper lines. If the rate of change is negative, the line is falling (decreasing) from left to right.

EXAMPLE

This example shows how to enter a list of two starting values so you can keep track of the term number in a sequence. The output is a list of paired values.

Guiding the Investigation

[ELL] *Heads for Flint* means travels in the direction of Flint. *Heads south* means travels south.

It may help to have six students go to the front of the classroom with signs that say *green minivan, Mackinac Bridge,* and so on, and walk through the problem before students begin working on it.

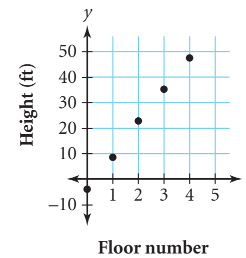
You can modify the context of the investigation to suit your geographical area by choosing cities and highways with approximately these distances, and creating your own transparency and worksheet.

The Fathom demonstration On the Road Again can be used to replace this investigation.

One Step
Pose the investigation problem. Ask students to answer the questions in as many ways as possible. Encourage use of recursive calculator routines and graphs. Students may say that obviously the minivan arrives before the pickup because it's going faster over the same distance. It may not be as obvious that the minivan will arrive before the sports car because the sports car has less distance to travel, but it's moving more slowly. Others may point out that the pickup will pass the sports car in less than 2 h because the situation is equivalent to the sports car sitting still 35 mi up the road and the pickup traveling at 18 mi/h (66 − 48). Encourage this kind of variety in approach.

Step 1 Throughout the investigation, encourage the use of units and dimensional analysis.

b. Let x represent the floor number and y represent the floor's height in feet. Mark a scale from 0 to 5 on the x-axis and −10 to 50 on the y-axis. Plot the data from the table. The graph starts at (0, −4) on the y-axis. The points appear to be in a line. It does not make sense to connect the points because it is not possible to have a decimal or fractional floor number.



c. The recursive routine generates the points (0, −4), (1, 9), (2, 22), . . . , (15, 191), (16, 204), ... The height of the 15th floor is 191 ft. The height of the 16th floor is 204 ft. So the 15th floor is the highest floor with a height less than 200 ft. No floor is exactly 200 ft high.

Notice that to get to the next point on the graph from any given point, move right 1 unit on the x-axis and up 13 units on the y-axis. The points you plotted in the example showed a **linear relationship** between floor numbers and their heights. In what other graphs have you seen linear relationships?

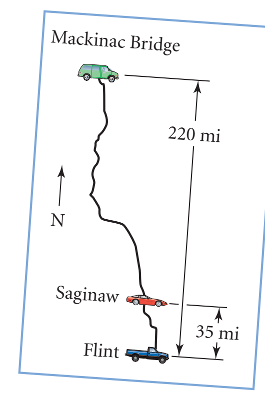
Investigation
On the Road Again

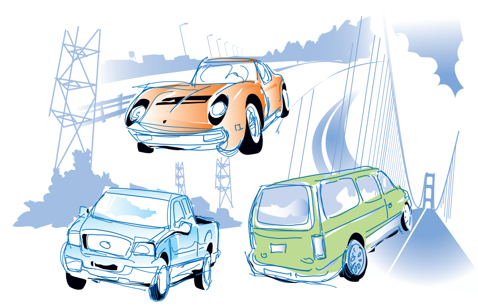
You will need

- the worksheet On the Road Again Grid

A green minivan starts at the Mackinac Bridge and heads south for Flint on Highway 75. At the same time, a red sports car leaves Saginaw and a blue pickup truck leaves Flint. The car and the pickup are heading for the bridge. The minivan travels 72 mi/h. The pickup travels 66 mi/h. The sports car travels 48 mi/h.

When and where will they pass each other on the highway? In this investigation you will learn how to use recursive sequences to answer questions like these.





Step 1 minivan: 1.2 mi/min; pickup: 1.1 mi/min; sports car: 0.8 mi/min

Step 1 Find each vehicle's average speed in miles per minute (mi/min).

Step 2 Write recursive routines to find each vehicle's distance from Flint at each minute. What are the real-world meanings of the starting value and the rule in each routine? Use calculator lists.

LESSON OBJECTIVES

- Graph scatter plots of recursive sequences
- Continue to explore the connection between graphs and tables and how they can be used to solve problems
- Build toward an introduction of the intercept form of a line

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number	✓	Problem Solving
✓	Algebra		Reasoning
✓	Geometry	✓	Communication
✓	Measurement	✓	Connections
	Data/Probability	✓	Representation

Step 3 Make a table to record the highway distance from Flint for each vehicle. After you complete the first few rows of data, change your recursive routines to use 10 min intervals for up to 4 h.

Highway Distance from Flint

Time (min)	Minivan (mi)	Sports car (mi)	Pickup (mi)
0	220	35	0
1	218.8	35.8	1.1
2	217.6	36.6	2.2
5	214	39	5.5
10	208	43	11
20	196	51	22

Procedure Note

After you enter the recursive routine into the calculator, press ENTER five or six times. Copy the data displayed on your calculator screen onto your table. Repeat this process.

Step 4 Define variables and plot the information from the table onto a graph. Mark and label each axis in 10-unit intervals, with time on the horizontal axis. Using a different color for each vehicle, plot its (*time, distance*) coordinates.

Step 4 Let x represent the time in minutes since the vehicles started their trips; let y represent the distance from Flint in highway miles.

Step 5 On the graph, do the points for each vehicle seem to fall on a line? Does it make sense to connect each vehicle's points in a line? If so, draw the line. If not, explain why not. Yes; a line through the points for each vehicle represents every possible instant of time.

Use your graph and table to find the answers for Steps 6–10.

Step 6 Where does the starting value for each routine appear on the graph? How does the recursive rule for each routine affect the points plotted?

Step 6 On the y-axis representing each vehicle's initial distance from Flint; the rules affect the steepness and direction of each line.

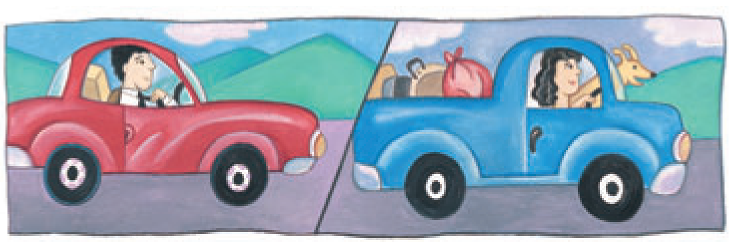
Step 7 Which line represents the minivan? How can you tell?

Step 8 Where are the vehicles when the minivan meets the first one headed north?

Step 8 The minivan meets the sports car about 110 mi from Flint just after 90 min. The pickup is about 100 mi from Flint.

Step 9 How can you tell by looking at the graph whether the pickup or the sports car is traveling faster? When and where does the pickup pass the sports car?

Step 9 The pickup is traveling faster because its line is steeper; it passes the sports car when their lines cross on the graph, after roughly 115 min.



Step 10 Which vehicle arrives at its destination first? How many minutes pass before the second and third vehicles arrive at their destinations? How can you tell by looking at the graph?

Step 10 The minivan arrives after approximately 185 min, the pickup 15 min later, and the sports car 45 min later.

Step 11 What assumptions about the vehicles are you making when you answer the questions in the previous steps? The vehicles travel at a constant speed, never speeding up, slowing down, or stopping.

Step 7 Be sure students go beyond an answer like "It's colored green." **[Ask]** "How did you know which line to color green?"

Step 7 The line going down from left to right; it starts 220 units above the origin on the y-axis and gets closer to the x-axis as time passes and the minivan gets closer to Flint.

Steps 8 and 9 **[Alert]** Students may have difficulty keeping in mind both variables being graphed. Students may say, or at least believe, something like "They're at the same place but not at the same time."

You can find very good approximations for the answers to the questions using either the table or the graph. Ask students to save their results if you are going to assign Lesson 5.2, Exercise 11.

Step 2 As needed, discuss how the different directions the vehicles are traveling affect whether you add or subtract in the recursive routine. Also discuss the difference between miles apart on the highway and miles apart in a straight line. The distances are all miles apart on the highway. Be ready to remind students how to enter recursive data into calculator lists. (See Calculator Note 0D.)

Step 2 To input the starting values in a calculator list for the times and the distances of the minivan, pickup, and sports car, respectively, press {0, 220, 0, 35}. To apply the rule, press {Ans (1) + 1, Ans(2) − 1.2, Ans(3) + 1.1, Ans(4) + 0.8}. The starting values represent the time in minutes and each vehicle's distance from Flint. The rule is to add or subtract the speed in miles per minute, depending on the vehicle's direction.

Step 3 Each student can choose a different car and generate the related sequence recursively. Remind students of the instant replay function (Calculator Note 2C). To modify the recursive routines to use 10 min intervals, you can recall the last entry and change it to read {Ans (1) + 10, Ans(2) − 12, Ans(3) + 11, Ans(4) + 8}.

Step 4 This is the most important phase of the investigation. Using different colors is important for differentiating the vehicles.

Step 5 The data will provide three linear patterns. Students should draw these three intersecting lines and justify connecting the points. This may be the first continuous graph students have constructed since Chapter 2. **[Alert]** Some students may interpret their coordinate graphs as the *paths* of the cars rather than as indicating the distance from a fixed point over time.

See page 723 for answers to Step 3.

Assessing Progress
Your observations should help you assess students' skill at finding averages, writing recursive routines, plotting points, calculating rates of change, and interpreting graphs.

Closing the Lesson

Quantities generated by a recursive additive sequence (with a constant rate of change) have a **linear relationship** because their graphs lie on a straight line. If the amount being added at each step is positive, the line will rise from left to right, with larger rates of change giving steeper lines. If the amount being added is negative, the line will fall from left to right.

BUILDING UNDERSTANDING

Students move among recursive sequences, linear relationships, and their graphs.

ASSIGNING HOMEWORK

Essential	2–4, 6, 7
Performance assessment	8, 10
Portfolio	6
Journal	7, 8
Group	6, 9
Review	1, 5, 11–14

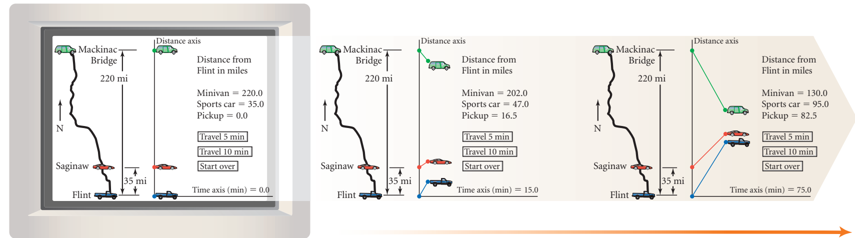
Helping with the Exercises

2a. {0.5, 1, 1.5, 2, 2.5, 3}; 0.5, Ans + 0.5
2b. {4, 3, 2, 1, 0}; 4, Ans − 1
2c. {−1, −0.75, −0.5, −0.25, 0, 0.25}; −1, Ans + 0.25
2d. {−1.5, 0, 1.5, 3}; −1.5, Ans + 1.5

Exercise 4 Be sure students note the instructions for 4a and b at the beginning of the exercise.

See page 723 for answers to Exercises 4a and b.

Step 12 Consider how to model this situation more realistically. What if the vehicles are traveling at different speeds? What if one driver stops to get gas or a bite to eat? What if the vehicles' speeds are not constant? Discuss how these questions affect the recursive routines, tables of data, and their graphs. If speeds are not constant, the points will not lie in a line and you would not be adding or subtracting the same number in the recursive routine. The lines would have horizontal pieces if drivers stop.



keymath.com/DA

[▶ You can use the **Dynamic Algebra Exploration** found at www.keymath.com/DA to further explore the situation described in the investigation. ◀]

EXERCISES

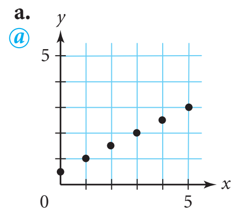
You will need your graphing calculator for Exercises **4–7** and **9.**

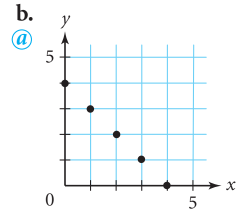
Practice Your Skills

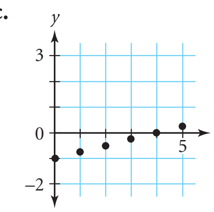
1. Decide whether each expression is positive or negative without using your calculator. Then check your answer with your calculator.

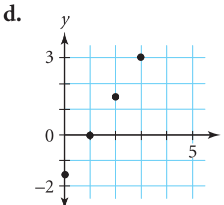
a. $-35(44) + 23$ negative; -1517
b. $(-14)(-36) - 32$ positive; 472
c. $25 - \frac{152}{12}$ positive; $12.\overline{3}$
d. $50 - 23(-12)$ positive; 326
e. $\frac{-12 - 38}{15}$ negative; $-3.\overline{3}$
f. $24(15 - 76)$ negative; -1464

2. List the terms of each number sequence of y-coordinates for the points shown on each graph. Then write a recursive routine to generate each sequence.

a. @


b. @


c.


d.


3. Make a table listing the coordinates of the points plotted in 2b and d.

2b.

x	y
0	4
1	3
2	2
3	1
4	0

2d.

x	y
0	−1.5
1	0
2	1.5
3	3

4. Plot the first five points represented by each recursive routine in 4a and b on separate graphs. Then answer 4c and d.

a. {0, 5} ENTER
{Ans(1) + 1, Ans(2) + 7} ENTER; ENTER, . . .

b. {0, −3} ENTER
{Ans(1) + 1, Ans(2) − 6} ENTER; ENTER, . . .

SHARING IDEAS

If some students had the idea that the graphs showed the paths of the vehicles, ask the class to critique that notion. This is a good place to ask questions beginning with "Are you saying . . . ?" in order to clarify ideas without having to tell much.

Have students present Steps 9–12. **[Ask]** "Are you saying that the relationship is linear because the rate of change is constant?" Elicit the idea that the amount being added at each step of the recursive routine is the rate of change.

If you have time, **[Ask]** "How do linear relationships relate to directly proportional quantities and direct variation?" [Directly proportional quantities have a linear relationship, but the opposite is not necessarily true. The graph of a direct variation is a straight line through the origin. The graph of a linear relationship between variables is also a straight line, but not necessarily through the origin.]

As a synonym for *steepness,* you might use the word *slope,* though its formal definition will not come until Chapter 4.

c. On which axis does each starting point lie? What is the x-coordinate of each starting point? the y-axis; 0

d. As the x-value increases by 1, what happens to the y-coordinates of the points in each sequence in 4a and b? @ In 4a, the y-coordinates increase by 7. In 4b, the y-coordinates decrease by 6.

5. The direct variation $y = 2.54x$ describes the relationship between two standard units of measurement where y represents centimeters and x represents inches.

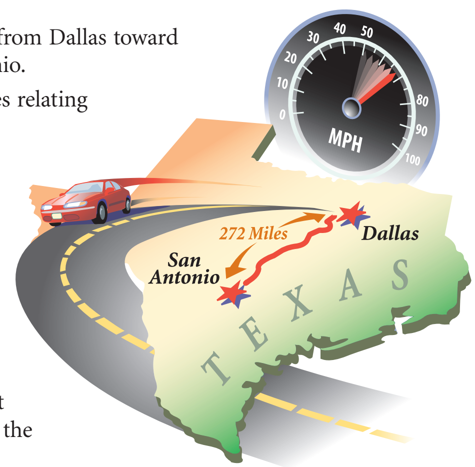
a. Write a recursive routine that would produce a table of values for any whole number of inches. Use a calculator list.

b. Use your routine to complete the missing values in this table.

Inches	Centimeters
0	0
1	2.54
2	5.08
14	35.56
17	43.18

Reason and Apply

6. **APPLICATION** A car is moving at a speed of 68 mi/h from Dallas toward San Antonio. Dallas is about 272 mi from San Antonio.



a. Write a recursive routine to create a table of values relating time to distance from San Antonio for 0 to 5 h in 1 h intervals.

b. Graph the information in your table.

c. What is the connection between your plot and the starting value in your recursive routine?

d. What is the connection between the coordinates of any two consecutive points in your plot and the rule of your recursive routine?

e. Draw a line through the points of your plot. What is the real-world meaning of this line? What does the line represent that the points alone do not?

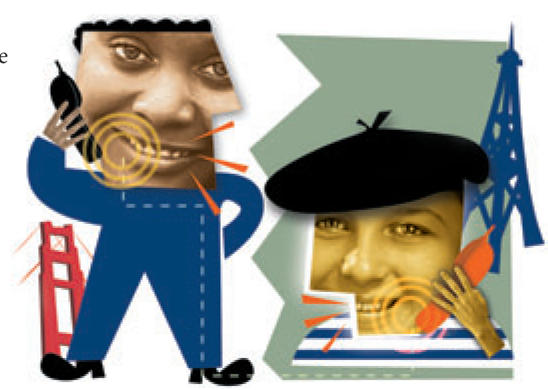
f. When is the car within 100 mi of San Antonio? Explain how you got your answer.

g. How long does it take the car to reach San Antonio? Explain how you got your answer.

7. **APPLICATION** A long-distance telephone carrier charges \$1.38 for international calls of 1 minute or less and \$0.36 for each additional minute.

a. Write a recursive routine using calculator lists to find the cost of a 7-minute phone call. @

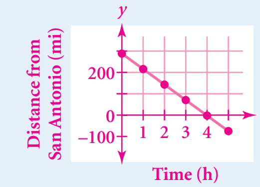
b. Without graphing the sequence, give a verbal description of the graph showing the costs for calls that last whole numbers of minutes. Include in your description all the important values you need in order to draw the graph.



5a. {0, 0} ENTER, {Ans(1) + 1, Ans(2) + 2.54}

6a. {0, 272} ENTER, {Ans(1) + 1, Ans(2) − 68} ENTER, ENTER, ENTER, ENTER, ENTER

6b and 6e.



6c. The starting value is the point (0, 272) on the graph.

6d. On the graph, you move right 1 unit and down 68 units to get from one point to the next. In the recursive routine, you add 1 to the first number and subtract 68 from the second number.

6e. This is a linear graph relating a distance to any time between 0 and 5 h. The line represents the distances at all possible times; points represent distances only at certain times.

6f. The car is within 100 mi of San Antonio after 2.53 h have elapsed. Explanations will vary. Graphically, it is the time after which the line crosses the horizontal line $y = 100$.

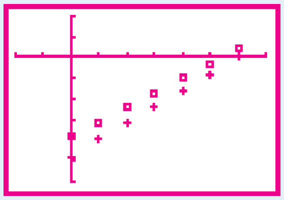
6g. The car takes 4 h to reach San Antonio. Answers will vary. The answer is the fourth entry in the table. Graphically, it is where the line crosses the x-axis.

7a. Possible answer: {1, 1.38} ENTER, {Ans(1) + 1, Ans(2) + 0.36} ENTER, ENTER, ... The recursive routine keeps track of time and cost for each minute. Apply the routine until you get {7, 3.54}. A 7 min call costs \$3.54.

7b. Possible answer: The graph should consist of points that lie on a line. It should include the point (1, 1.38). Each subsequent point should be 1 unit to the right and \$0.36 higher than the point before it.

Exercise 8 Discuss how measuring depth on the bow, bridge, or stern of a submarine as it is surfacing affects data.

8a.



[−10, 35, 5, −60, 20, 10]

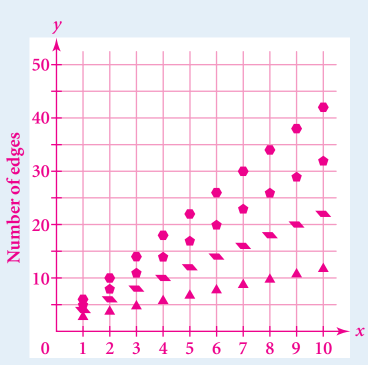
8b. The points for each submarine appear to lie on a line; the USS *Dallas* surfaces at a faster rate.

Exercise 9 Suggest that students calculate the perimeter by treating each edge as 1 unit long.

9b. Number of tiles: The starting value is 1; the rule is add 1.
Triangle: The starting value is 3; the rule is add 1.
Rhombus: The starting value is 4; the rule is add 2.
Pentagon: The starting value is 5; the rule is add 3.
Hexagon: The starting value is 6; the rule is add 4.
To generate the sequences for all tiles simultaneously, enter {1, 3, 4, 5, 6} and {Ans(1) + 1, Ans(2) + 1, Ans(3) + 2, Ans(4) + 3, Ans(5) + 4}.

9c. triangle: 52; rhombus: 102; pentagon: 152; hexagon: 202

9d.



8. These tables show the changing depths of two submarines as they come to the surface.

USS *Alabama*

Time (s)	0	5	10	15	20	25	30
Depth (ft)	−38	−31	−24	−17	−10	−3	4

USS *Dallas*

Time (s)	0	5	10	15	20	25	30
Depth (ft)	−48	−40	−32	−24	−16	−8	0

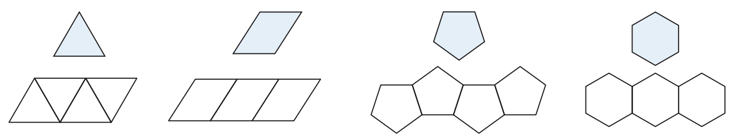
a. Graph the data from both tables on the same set of coordinate axes.

b. Describe what you found by graphing the data. How are the graphs the same? How are they different?

c. Does it make sense to draw a line through each set of points? Explain what these lines mean. **Yes; each line means that any time in this range corresponds to depth below the surface.**

d. What is the real-world meaning of the point (30, 4) for the USS *Alabama*? **The submarine's nose rises slightly above the water when surfacing.**

9. Each geometric design is made from tiles arranged in a row.



Triangle Rhombus Pentagon Hexagon

a. Make a table like the one shown. Find the number of tile edges on the perimeter of each design, and fill in ten rows of the table. Look for patterns as you add more tiles. ⓗ

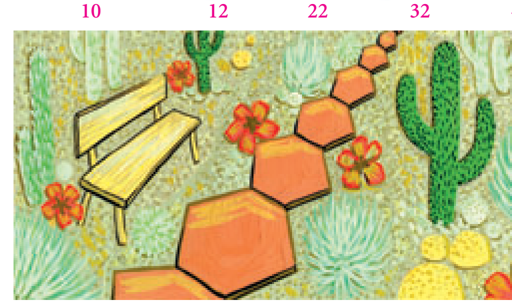
Tile Edges on the Perimeter

Number of tiles	Triangle	Rhombus	Pentagon	Hexagon
1	3	4	5	6
2	4	6	8	10
3	5	8	11	14
4	6	10	14	18
⋮				
10	12	22	32	42

b. Write a recursive routine to generate the values in each table column.

c. Find the perimeter of a 50-tile design for each shape.

d. Draw four plots on the same coordinate axes using the information for designs of one to ten tiles of each shape. Use a different color for each shape. Put the number of tiles on the horizontal axis and the number of edges on the vertical axis. Label and scale each axis.



e. Compare the four scatter plots. How are they alike, and how are they different?

f. Would it make sense to draw a line through each set of points? Explain why or why not. ⓗ **No; there must be a whole number of tiles and a whole number of edges.**

9e. The points of each graph appear to lie on a line, and each graph starts at 1; the graphs increase in steepness from the triangle tile to the hexagon tile.

10. A bicyclist, 1 mi (5280 ft) away, pedals toward you at a rate of 600 ft/min for 3 min. The bicyclist then pedals at a rate of 1000 ft/min for the next 5 min.

a. Describe what you think the plot of (*time, distance from you*) will look like. @

b. Graph the data using 1 min intervals for your plot. @

c. Invent a question about the situation, and use your graph to answer the question.

Review

11b.

Operations	Undo operations	Results
		$x = 3.4$
$-$ (2.8)	$+$ (2.8)	0.6
$\cdot$ (3.2)	/ (3.2)	1.92
$+$ (5.4)	$-$ (5.4)	7.32
/ (1.2)	$\cdot$ (1.2)	6.1
$-$ (2.3)	$+$ (2.3)	3.8

2.8 **11.** Consider the expression

$$\frac{5.4 + 3.2(x - 2.8)}{1.2} - 2.3$$

a. Use the order of operations to find the value of the expression if $x = 7.2$. $13.9\overline{3}$

b. Set the expression equal to 3.8. Solve for x by undoing the sequence of operations you listed in 11a. $x = 3.4$

2.8 **12.** Isaac learned a way to convert from degrees Celsius to Fahrenheit. He adds 40 to the Celsius temperature, multiplies by 9, divides by 5, and then subtracts 40.

a. Write an expression for Isaac's conversion method. @ $\frac{9(C + 40)}{5} - 40$

b. Write the steps to convert from Fahrenheit to Celsius by undoing Isaac's method. @

c. Write an expression for the conversion in 12b. $\frac{5(F + 40)}{9} - 40$

2.3 **13. APPLICATION** Karen is a U.S. exchange student in Austria. She wants to make her favorite pizza recipe for her host family, but she needs to convert the quantities to the metric system. Instead of using cups for flour and sugar, her host family measures dry ingredients in grams and liquid ingredients in liters. Karen has read that 4 cups of flour weigh 1 pound.

In her dictionary, Karen looks up conversion factors and finds that 1 ounce $\approx$ 28.4 grams, 1 pound $\approx$ 454 grams, and 1 cup $\approx$ 0.236 liter.

a. Karen's recipe calls for $\frac{1}{2}$ cup water and $1\frac{1}{2}$ cups flour. Convert these quantities to metric units.

b. Karen's recipe says to bake the pizza at 425°. Convert this temperature to degrees Celsius. Use your work in Exercise 12 to help you. $218.\overline{3}$, or about 220°C

1.6 **14.** Draw and label a coordinate plane with each axis scaled from -10 to 10.

a. Represent each point named with a dot, and label it using its letter name.

$A(3, -2)$ $B(-8, 1.5)$ $C(9, 0)$ $D(-9.5, -3)$ $E(7, -4)$

$F(1, -1)$ $G(0, -6.5)$ $H(2.5, 3)$ $I(-6, 7.5)$ $J(-5, -6)$

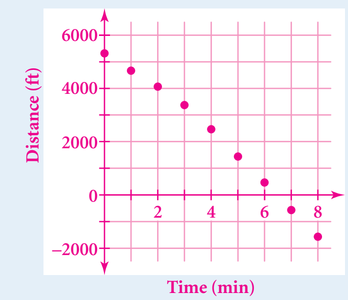
b. List the points in Quadrant I, Quadrant II, Quadrant III, and Quadrant IV. Which points are on the x-axis? Which points are on the y-axis?

c. Explain how to tell which quadrant a point will be in by looking at the coordinates. Explain how to tell if a point lies on one of the axes.

10a. Answers will vary. The graph starts at (0, 5280). The points (0, 5280), (1, 4680), (2, 4080), and (3, 3480) will appear to lie on a line. From (3, 3480) to (8, -1520), the points will appear to lie on a steeper line. The bicyclist ends up 1520 ft past you.

10b.

Bicyclist

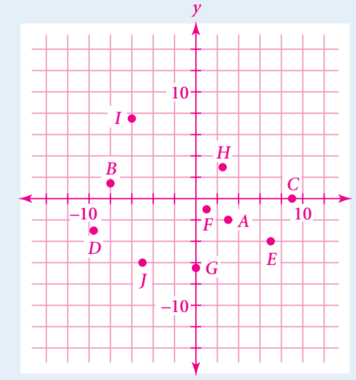


10c. Sample answer: What place on the graph shows when the bicyclist passes you? The answer is on the x-axis between 6 and 7 min.

12b. Add 40, multiply by 5, divide by 9, then subtract 40.

13a. 0.118, about $\frac{1}{8}$ L water; 170.25, about 170 g flour

14a.



14b. Quad I: H; Quad II: B, I; Quad III: D, J; Quad IV: A, E, F; x-axis: C; y-axis: G

14c. Sample answer: If the coordinates are both 0, then the point is on the origin. If the x-coordinate is 0, then the point is on the y-axis. If the y-coordinate is 0, then the point is on the x-axis.

If the first coordinate is positive, then the point will be in Quadrant I or IV. To tell which quadrant, look at the y-coordinate. If the y-coordinate is positive, the point is in Quadrant I. If the y-coordinate is negative, the point is in Quadrant IV.

If the first coordinate is negative, then the point will be in Quadrant II or III. To tell which quadrant, look at the y-coordinate. If the y-coordinate is positive, the point is in Quadrant II. If the y-coordinate is negative, the point is in Quadrant III.

LESSON
3.3

PLANNING

LESSON OUTLINE

One day:

30 min Investigation
10 min Example A
5 min Example B
5 min Exercises

MATERIALS

- 4 m measuring tapes, metersticks, or ropes
- motion sensors
- stopwatches or watches with second hands
- Calculator Note 3B

TEACHING

Becoming more familiar with time-distance graphs helps deepen students' understanding of graphs and rates of change.

One Step

Post a plot with a large circle in the first quadrant. **[Ask]** "How would a walker walk to produce this graph?" After some students respond that the walker should simply walk in a circle, label the axes with *time* and *distance.* Help students see that a single walker can't produce such a graph without being at two places at one time. Ask for other examples of plots that are impossible to make with a walker and motion sensor.

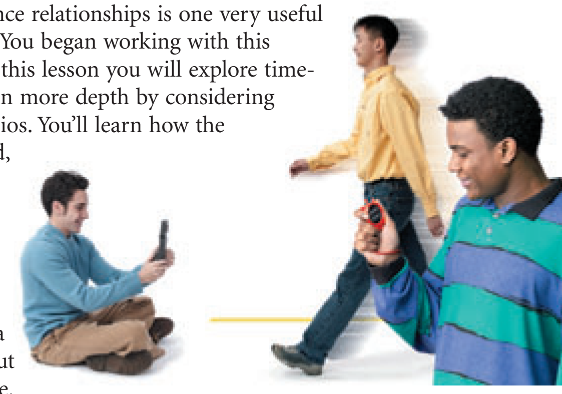
Guiding the Investigation

You might modify this investigation to use finger walking and rulers. Use the same numbers, but substitute inches for meters as the unit of measure.

LESSON
3.3

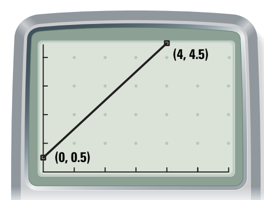
Time-Distance Relationships

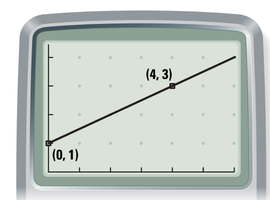
Modeling time-distance relationships is one very useful application of algebra. You began working with this topic in Lesson 3.2. In this lesson you will explore time-distance relationships in more depth by considering various walking scenarios. You'll learn how the starting position, speed, direction, and final position of a walker influence a graph and an equation.



The (*time, distance*) graphs below provide a lot of information about the "walks" they picture. The fact that the lines are straight and increasing means that both walkers are moving away from the motion sensor at a steady rate. The first walker starts 0.5 meter from the sensor, whereas the second walker starts 1 meter from the sensor. The first graph pictures a walker moving $4.5 - 0.5 = 4$ meters in $4 - 0 = 4$ seconds, or 1 meter per second. The second walker covers $3 - 1 = 2$ meters in $4 - 0 = 4$ seconds, or 0.5 meter per second.

In this investigation you'll analyze time-distance graphs, and you'll use a motion sensor to create your own graphs.



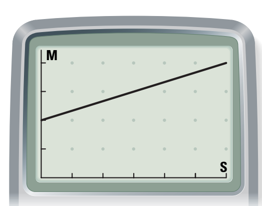


Investigation
Walk the Line

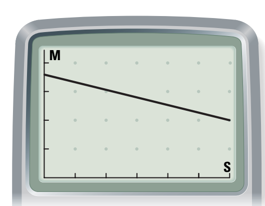
You will need

- a 4-meter measuring tape or four metersticks per group
- a motion sensor
- a stopwatch or watch that shows seconds

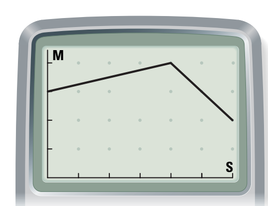
Imagine that you have a 4-meter measuring tape positioned on the floor. A motion sensor measures your distance from the tape's 0-mark as you walk, and it graphs the information. On the calculator graphs shown here, the horizontal axis shows time from 0 to 6 seconds and the vertical axis shows distance from 0 to 4 meters.



a.



b.



c.

LESSON OBJECTIVES

- Explore time-distance relationships
- Write walking instructions or act out walks for a given graph
- Sketch graphs based on given walking instructions or table data
- Use an electronic data collection device, motion sensor, and graphing calculator to collect and graph data

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
	Geometry	✓	Communication
✓	Measurement	✓	Connections
	Data/Probability	✓	Representation

Step 1 | Write a set of walking instructions for each graph. Tell where the walk begins, how fast the person walks, and whether the person walks toward or away from the motion sensor located at the 0-mark.

Step 2 | Graph a 6-second walk based on each set of walking instructions or data.

a. Start at the 2.5-meter mark and stand still.

b. Start at the 3-meter mark and walk toward the sensor at a constant rate of 0.4 meter per second.

c.

Time (s)	0	1	2	3	4	5	6
Distance (m)	0.8	1.0	1.2	1.4	1.6	1.8	2.0

Step 3 | Write a recursive routine for the table in Step 2c. The starting value is 0.8; the rule is add 0.2.

Steps 4 and 5 Students should discuss the difficulty of walking at a constant speed and of changing speed or direction at a specific instant in time. The coach needs to give good directions.

For the next part of the investigation, you will need a graphing calculator and a motion sensor. Your group will need a space about 4 meters long and 1.5 meters wide (13 feet by 5 feet). Tape to the floor a 4-meter measuring tape or four metersticks end-to-end. Assign these tasks among your group members: walker, motion-sensor holder, coach, and timer.

Step 4 | Your group will try to create the graph shown in Step 1, graph a. Remember that you wrote walking directions for this graph. Use your motion sensor to record the walker's motion. [▶ See **Calculator Note 3B** for help using the motion sensor. ◀] After each walk, discuss what you could have done to better replicate the graph. Repeat the walk until you have a good match for graph a.

Step 5 | Rotate jobs, and repeat Step 4 to model graphs b and c from Step 1 and the three descriptions from Step 2.

Using motion-sensor technology in the investigation, you were able to actually see how accurately you duplicated a given walk. The next examples will provide more practice with time-distance relationships.

EXAMPLE A

a. Graph a walk from the set of instructions "Start at the 0.5-meter mark and walk at a steady 0.25 meter per second for 6 seconds."

b. Write a set of walking instructions based on the table data, and then sketch a graph of the walk.

Time (s)	0	1	2	3	4	5	6
Distance (m)	4.0	3.6	3.2	2.8	2.4	2.0	1.6

Step 1 If students need a hint, suggest that because the vertical axis measures distance, they can answer the question "How far does the walker start from the zero mark?" by considering the vertical axis. If students have trouble answering "How fast?" you can give a hint by pointing to graph a. **[Ask]** "How long did it take the walker to go 1 m from his or her starting point?" Some students may have trouble seeing that the walker ever walks toward the 0 mark because the graphs appear to proceed away from the start. Ask them to look at graph b and read off "meters" at the start and finish. Graph c can't be defined with just one rule.

Steps 4 and 5 Students will gain more if they are physically involved, even if minimally. If you have limited classroom space or equipment, you might have different groups demonstrate walks one at a time. You might select students to demonstrate two or three "walks" before the others start. However you do it, try to have each student take on each role—walker, holder, coach, and timer—for at least one walk. A group need not have a coach if there are too few students.

If you're not using motion sensors, the holder records in a table the walker's distance from the 0 m mark each second. The timer begins timing when the coach tells the walker to start and counts the seconds aloud.

If you are using motion sensors, the holder should hold the calculator and start collecting data on the coach's command. (See Calculator Note 3B.) The holder should hold the motion sensor chest high, keeping it level and aimed directly at the walker. Timing begins when the coach tells the walker to start. The coach makes helpful comments to guide the walker.

Step 5 Groups should discuss what they could have done to match more accurately the given graph, table, or instructions.

EXAMPLE A

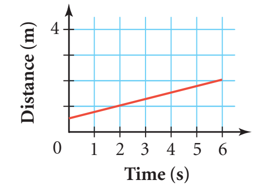
This example gives students more practice in graphing walking instructions, including some derived from a data table. As in the investigation, the rates are constant and the graphs are straight lines, slanting both upward and downward.

See page 723 for answers to Steps 1 and 2.

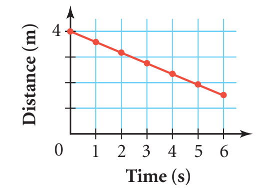
► Solution

Think about where the walker starts and how much distance he or she will cover in a given amount of time.

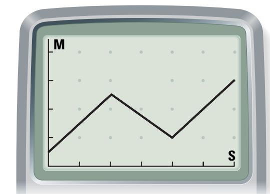
a. Walking at a steady rate of 0.25 meter per second for 6 seconds means the walker will move 0.25 m/s · 6 s = 1.5 m. The walker starts at 0.5 m and ends at 0.5 + 1.5 = 2 m.



b. Walking instructions: "Start at the 4-meter mark and walk toward the sensor at 0.4 meter per second." You can graph this walk by plotting the data points given.



EXAMPLE B Write a set of walking instructions for this graph:



► Solution

Start at the 0.5 m mark and walk away from the motion sensor at 1 m/s for 2 s. Then walk toward the sensor at $\frac{3}{4}$ m/s for 2 s. Then walk away from the sensor at 1 m/s for 2 s.

EXERCISES

Practice Your Skills

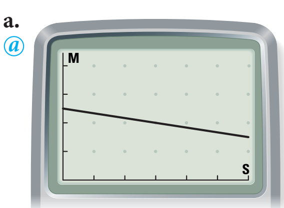
1. Write a recursive routine for the table in Example A, part b. @
{0, 4.0} and {Ans(1) + 1, Ans (2) − 0.4}

2. Sketch a graph of a walk starting at the 1-meter mark and walking away from the sensor at a constant rate of 0.5 meter per second.

3. Write a set of walking instructions and sketch a graph of the walk described by {0, 0.8} and {Ans(1) + 1, Ans(2) + 0.2}. @

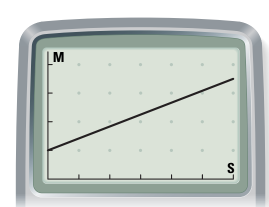
4. Describe the walk shown in each graph. Include where it started and how quickly and in what direction the walker moved.

a. @



The walker starts 2.5 m away from the motion sensor and walks toward it very slowly at a rate of 1 m in 6 s.

b.



The walker starts 1 m away from the motion sensor and walks away from it at a rate of 2.5 m in 6 s.

EXAMPLE B

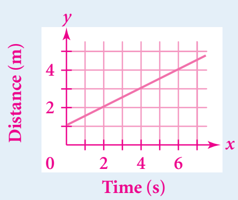
This example is for students who need further experience with writing walking instructions from a straight-line graph.

SHARING IDEAS

Have students present a variety of graphs and walking directions, including recursive routines. If a group has a graph that isn't a straight line, ask the group to present that graph as a springboard into a discussion of how speed affects a graph.

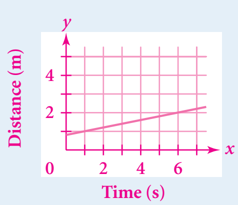
[Ask] "Can you tell from the graph whether the walker is moving toward or away from the motion sensor?" Elicit the idea that a horizontal line graph represents movement in neither direction and that a vertical line graph is impossible.

2.



[Ask] "Is it clear how a recursive routine relates to the graph?" You need not mention the term *slope* or give a formal definition, but relate the additive constant to the speed of walking and the steepness of the line.

3. Start at the 0.8 m mark and walk away from the sensor at a constant rate of 0.2 m/s.



Closing the Lesson

The main point of this lesson is that motion at a constant speed is graphed with a straight line. If the walker is moving away from the motion sensor, so that distance is increasing, the line rises from left to right; if the walker is moving toward the sensor, so that distance is decreasing, the line falls from left to right; if the walker is motionless, the line is horizontal.

The speed of the walker is related to the steepness of the line and to the additive constant in a recursive routine. The higher the speed, the steeper the graph and the farther the additive constant is from 0.

5. Describe the walk represented by the data in each table.

a. ⓐ

Time (s)	Distance (m)
0	6
1	5.8
2	5.6
3	5.4
4	5.2
5	5.0
6	4.8

The walker starts 6 m away from the motion sensor and walks toward it at a rate of 0.2 m/s for 6 s.

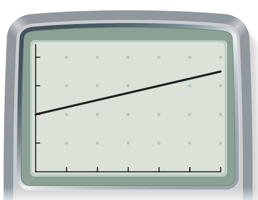
b.

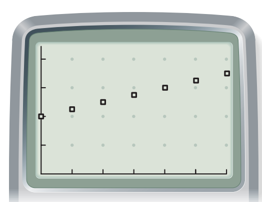
Time (s)	Distance (m)
0	1
1	1.6
2	2.2
3	2.8
4	3.4
5	4.0
6	4.6

The walker starts 1 m away from the motion sensor and walks away from it at a rate of 0.6 m/s for 6 s.

Reason and Apply

6. Which graph better represents a walk in which the walker starts 2 m from the motion sensor and walks away from it at a rate of 0.25 m/s for 6 s? Explain.

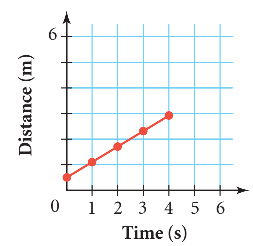




The first graph, which shows a line, because the walk is a continuous process; the walker is somewhere at every possible time in the 6 s.

7. At what rate in ft/s would you walk so that you were moving at a constant speed of 1 mi/h? ⓗ

8. The time-distance graph shows Carol walking at a steady rate. Her partner used a motion sensor to measure her distance from a given point.



a. According to the graph, how much time did Carol spend walking? 4 s

b. Was Carol walking toward or away from the motion sensor? Explain your thinking. ⓐ Away; the distance is increasing.

c. Approximately how far away from the motion sensor was she when she started walking? approximately 0.5 m

d. If you know Carol is 2.9 m away from the motion sensor after 4 s, how fast was she walking? ⓗ $\frac{2.9-0.5}{4} = 0.6$ m/s

e. If the equipment will measure distances only up to 6 m, how many seconds of data can be collected if Carol continues walking at the same rate? ⓐ

f. Looking only at the graph, how do you know that Carol was neither speeding up nor slowing down during her walk? ⓐ The graph is a straight line.

9. Draw a scatter plot on your paper picturing (*time, distance*) at 1 s intervals if you start timing Carol's walk as she walks toward her partner starting at a distance of 5.9 m and moving at a constant speed of 0.6 m/s.

BUILDING UNDERSTANDING

In these exercises students describe walks using multiple representations, including graphs, tables, recursive rules, and walking instructions.

ASSIGNING HOMEWORK

Essential	1–5, 9
Performance assessment	6, 8, 10, 11, 13
Portfolio	10
Journal	6, 12
Group	12, 13
Review	7, 14–16

Helping with the Exercises

Exercise 6 **[Alert]** Students may be confused by the dots. **[Ask]** "Where is the walker after 2.5 seconds? After 4.25 seconds?"

Exercise 7 If students are stuck, you might refer them to Lesson 2.8, Exercise 5.

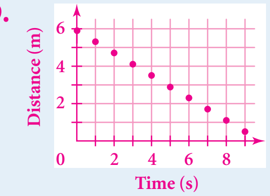
7. Convert 1 mi/h to ft/s:

$$\frac{1\text{ mi}}{1\text{ h}} \cdot \frac{1\text{ h}}{60\text{ min}} \cdot \frac{1\text{ min}}{60\text{ s}} \cdot \frac{5280\text{ ft}}{1\text{ mi}} = 1.4\overline{6}\text{ ft/s}$$

8e. $\frac{5.5\text{ m}}{0.6\text{ m/s}} = 9.1\overline{6}$ s, or approximately 9 s

Exercise 9 As needed, remind students that a scatter plot is simply a collection of dots.

9.



10. Describe how the rate affects the graph of each situation.

a. The graph of a person walking toward a motion sensor. @ The rate is negative, so the line slopes down to the right.

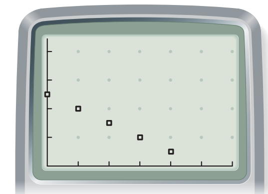
b. The graph of a person standing still. The rate is neither negative nor positive, it is zero, so the line is horizontal.

c. The graph of a person walking slowly.
The line is not very steep.

11. Match each calculator Answer routine to a graph.

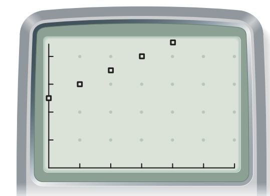
ii **a.** 2.5 ENTER
Ans + 0.5, ENTER, ENTER, . . . @

i.



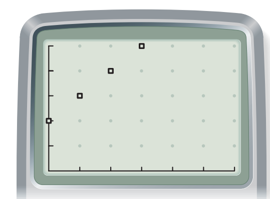
iv **b.** 1.0 ENTER
Ans + 1.0, ENTER, ENTER, . . .

ii.



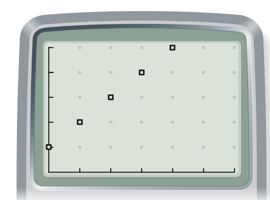
iii **c.** 2.0 ENTER
Ans + 1.0, ENTER, ENTER, . . .

iii.



i **d.** 2.5 ENTER
Ans − 0.5, ENTER, ENTER, . . .

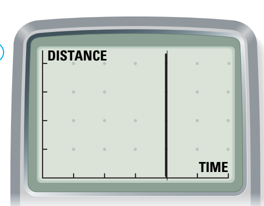
iv.



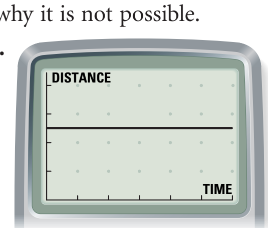
12. Describe how you would instruct someone to walk the line $y = x$, where x is measured in seconds and y is measured in feet. Describe how to walk the line $y = x$, where x is measured in seconds and y is measured in meters. Which line represents a faster rate? Explain.

13. For each situation, determine if it is possible to collect such walking data and either describe how to collect it or explain why it is not possible.

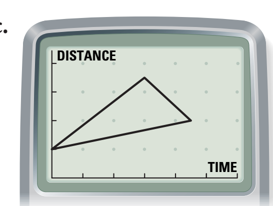
a. @



b.



c.



12. Start walking at the 0 mark when the sensor starts and walk 1 ft every second. Start walking at the 0 mark when the sensor starts and walk 1 m every second. 1 m/s is a faster rate, because more distance is covered per second.

13a. Not possible; the walker would have to be at more than one distance from the sensor at the 3 s mark.

13b. Possible; the walker simply stands still about 2.5 m from the sensor.

13c. Not possible; the walker can't be in two places at any given time.

Review

2.1 **14.** Solve each proportion for x.

a. $\frac{x}{3} = \frac{7}{5}$ $\quad x = \frac{21}{5}$, or 4.2

b. $\frac{2}{x} = \frac{9}{11}$ ⓐ $\quad x = \frac{22}{9}$, or 2.4

c. $\frac{x}{c} = \frac{d}{e}$ $\quad x = \frac{cd}{e}$

2.3 **15.** On his Man in Motion World Tour in 1987, Canadian Rick Hansen wheeled himself 24,901.55 miles to support spinal cord injury research and rehabilitation, and wheelchair sport. He covered 4 continents and 34 countries in two years, two months, and two days. Learn more about Rick's journey with the link at www.keymath.com/DA .

a. Find Rick's average rate of travel in miles per day. (Assume there are 365 days in a year and 30.4 days in a month.) ⓗ

b. How much farther would Rick have traveled if he had continued his journey for another $1\frac{1}{2}$ years?

c. If Rick continued at this same rate, how many days would it take him to travel 60,000 miles? How many years is that?


Photo courtesy of The Rick Hansen Institute

China was one of the many countries through which Rick Hansen traveled during the Man in Motion World Tour.

2.3 **16.** **APPLICATION** Nicholai's car burns 13.5 gallons of gasoline every 175 miles.

a. What is the car's fuel consumption rate? ⓗ $\sim$ 13 mi/gal or 0.077 gal/mi

b. At this rate, how far will the car go on 5 gallons of gas? 65 mi

c. How many gallons does Nicholai's car need to go 100 miles? 7.7 gal

Exercise 15 [Alert] Students may be misled by the extra information about continents and countries. The rate needed for 15c is the reciprocal of the rate needed for 15a and 15b.

15a. $\frac{24{,}901.55\text{ mi}}{(2 \cdot 365 + 2 \cdot 30.4 + 2)\text{ days}}$

≈ 31.4 mi/day

15b. $\frac{31.4\text{ mi}}{1\text{ day}} \cdot \frac{(1.5 \cdot 365)\text{ days}}{1}$

$\approx 17{,}191.5$ mi

15c. $\frac{31.4\text{ mi}}{1\text{ day}} = \frac{60{,}000\text{ mi}}{t}$;

$t \approx 1{,}911$ days, or more than 5 yr

project

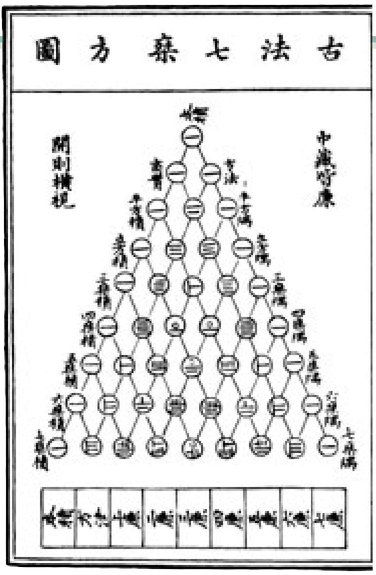
PASCAL'S TRIANGLE

The first five rows of Pascal's triangle are shown.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

The triangle can be generated recursively. The sides of the triangle are 1's, and each number inside the triangle is the sum of the two diagonally above it.

Complete the next five rows of Pascal's triangle. Research its history and practical application. What is the connection between Sierpiński's triangle and Pascal's triangle? Can you find the sequence of triangular numbers in Pascal's triangle? What is its connection to the Fibonacci number sequence? Present your findings in a paper or a poster.



What became known as Pascal's triangle was first published in *Siyuan yujian xicao* by Zhu Shijie in 1303. This ancient version actually has one error. Can you find it?

The fourth number in row 8 should be 35.

Pascal's Triangle Project

The next four rows are:
1, 6, 15, 15, 6, 1
1, 7, 21, 35, 35, 21, 7, 1
1, 8, 28, 56, 70, 56, 28, 8, 1
1, 9, 36, 84, 126, 126, 84, 36, 9, 1

Fibonacci Numbers

1 3 3 1 21
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Sierpiński's Triangle

- If the odd numbers in Pascal's triangle are colored in, the even numbers are left uncolored, and the triangle is extended infinitely, then it becomes a Sierpiński triangle.

Supporting the project

MOTIVATION

This pattern of numbers is named for a 17th-century mathematician. He used this pattern of numbers extensively in his study of probability. Was that the beginning of the history of this number pattern? (See above for the next four rows.)

OUTCOMES

- The recursive rule includes a starting 1 and this rule: If a number is at the end of a row, it's 1, and if it's not, then it's the sum of the two numbers diagonally above it.
- The report includes history going back to the ancient Chinese civilization.
- The triangular numbers (1, 3, 6, 10, 15, . . .) are in the third diagonal.
- The Fibonacci numbers (1, 1, 2, 3, 5, 8, . . .) are sums of the numbers on diagonals described by "start with a 1 on the left, go over one and a half numbers and up to the next row, follow that diagonal, adding the numbers." (For example, 1 + 3 + 1 = 5; 1 + 4 + 3 = 8; 1 + 5 + 6 + 1 = 13.)
- Sierpiński's triangle (see above)

LESSON

3.4

PLANNING

LESSON OUTLINE

One day:

25 min Investigation

5 min Sharing

10 min Examples

5 min Closing

5 min Exercises

MATERIALS

- Calculator Note 1J
- Fathom demonstration Working Out, *optional*

TEACHING

[Language] Define *metabolism* as the physical and chemical processes that maintain the body, for instance, turning food into energy.

In this lesson students make the transition from using recursive routines to writing linear equations in intercept form. The intercept form is like the well-known slope-intercept form except that the order is different: $y = a + bx$. This order emphasizes starting with the number a and adding the number b repeatedly (x times).

Guiding the Investigation

The Fathom demonstration Working Out can replace this investigation.

One Step

Pose this problem: "Manisha starts her exercise routine by jogging to the gym, which burns 215 calories. At the gym she pedals a stationary bike, burning 3.8 calories per minute. How long will it take her to burn a total of 538 calories?" Encourage students to

LESSON

3.4

Linear Equations and the Intercept Form

So far in this chapter you have used recursive routines, graphs, and tables to model linear relationships. In this lesson you will learn to write **linear equations** from recursive routines. You'll begin to see some common characteristics of linear equations and their graphs, starting with the relationship between exercise and calorie consumption.

Different physical activities cause people to burn calories at different rates depending on many factors such as body type, height, age, and metabolism. Coaches and trainers consider these factors when suggesting workouts for their athletes.

Investigation
Working Out with Equations

Manisha starts her exercise routine by jogging to the gym. Her trainer says this activity burns 215 calories. Her workout at the gym is to pedal a stationary bike. This activity burns 3.8 calories per minute.

First you'll model this scenario with your calculator.

Manisha's Workout

Pedaling time (min) x	Total calories burned y
0	215
1	218.8
2	222.6
20	291
30	329
45	386
60	443

Step 1 Use calculator lists to write a recursive routine to find the total number of calories Manisha has burned after each minute she pedals the bike. Include the 215 calories she burned on her jog to the gym.

Step 1 $\{0, 215\}$ ENTER, $\{\text{Ans}(1) + 1, \text{Ans}(2) + 3.8\}$ ENTER, ENTER . . .

Step 2 Copy and complete the table using your recursive routine.

Step 3 After 20 minutes of pedaling, how many calories has Manisha burned? How long did it take her to burn 443 total calories?

Step 3 291 calories; 60 min

LESSON OBJECTIVES

- Write a linear equation in intercept form given a recursion routine, a graph, or data
- Learn the meaning of y-intercept for a linear equation in intercept form

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
	Geometry	✓	Communication
	Measurement	✓	Connections
	Data/Probability	✓	Representation

Next you'll learn to write an equation that gives the same values as the calculator routines.

Step 4 $215 + 3.8(20) = 291$

Step 4 Write an expression to find the total calories Manisha has burned after 20 minutes of pedaling. Check that your expression equals the value in the table.

Step 5 $215 + 3.8(38) = 359.4$ calories; you don't need to calculate the previous terms in the sequence or create a table to find the answer.

Step 5 Write and evaluate an expression to find the total calories Manisha has burned after pedaling 38 minutes. What are the advantages of this expression over a recursive routine?

Step 6 Let x represent the pedaling time in minutes, and let y represent the total number of calories Manisha burns. Write an equation relating time to total calories burned. $y = 215 + 3.8x$

Step 7 sample checks: $215 + 3.8(1) = 218.8$; $215 + 3.8(60) = 443$

Step 7 Check that your equation produces the corresponding values in the table.

Now you'll explore the connections between the linear equation and its graph.

Step 8 The x-axis represents every instant of time, and the y-axis represents every fraction of calories burned. This graph models a continuous linear relationship. See graph below.

Step 8 Plot the points from your table on your calculator. Then enter your equation into the Y= menu. Graph your equation to check that it passes through the points. Give two reasons why drawing a line through the points realistically models this situation. [See **Calculator Note 1J** to review how to plot points and graph an equation.]

Step 9 Substitute 538 for y in your equation to find the elapsed time required for Manisha to burn a total of 538 calories. Explain your solution process. Check your result. $538 = 215 + 3.8x$; $x = 85$. Check: $215 + 3.8(85) = 538$.

Step 10 How do the starting value and the rule of your recursive routine show up in your equation? How do the starting value and the rule of your recursive routine show up in your graph? When is the starting value of the recursive routine also the value where the graph crosses the y-axis?

The equation for Manisha's workout shows a linear relationship between the total calories burned and the number of minutes pedaling on the bike. You probably wrote this linear equation as

$$y = 215 + 3.8x \quad \text{or} \quad y = 3.8x + 215$$

The form $y = a + bx$ is the **intercept form.** The value of a is the ***y*-intercept,** which is the value of y when x is zero. The intercept gives the location where the graph crosses the y-axis. The number multiplied by x is b, which is called the **coefficient** of x.

Step 8

Manisha's Workout

Step 10 In the equation $y = 215 + 3.8x$, the starting value is 215 and the rule to add shows up as the coefficient of x. The starting value is the y-intercept. The rule add 3.8 gives the steepness of the line. The starting value of the recursive routine is the y-intercept only when the starting value of x is zero.

use recursive routines, scatter plots, linear equations, graphs, and calculator tables to solve the problem. Be sure Sharing includes ideas about equations.

Step 1 **[Language]** If students ask, tell them that a *calorie* is the amount of heat energy needed to warm 1 g of water 1°C.

Step 4 As needed, suggest that students create an expression in words before they use symbols. Be sure they haven't forgotten to include the initial 215 calories.

Step 9 Allow any legitimate method for solving the equation, but insist on a good explanation. **[Alert]** Be especially wary of any student impulses to move numbers from one side of the equation to the other.

SHARING IDEAS

Ask several students to share their solution methods for Step 9.

At an appropriate time, introduce the term *intercept form* for the equation $y = a + bx$ and the related terms *y-intercept* and *coefficient of x*. **[Ask]** "Is the equation $y = bx + a$ equivalent to the intercept form?" [Yes] "Are equations $y = ax + b$ and $y = mx + b$ (often called the *slope-intercept form*) also equivalent?" [Yes] Substitute numbers for a, b, and m as needed. These questions may allow students to see that the letters a, b, and m represent constants in particular equations, whereas the letters x and y represent variables. **[Ask]** "Is the equation $y = a - bx$ equivalent to $y = bx - a$?" [No; $y = a - bx$ is equivalent to $y = -bx + a$. It is very important to keep the signs consistent.]

[Ask] "How are linear equations related to other equations you have seen in this course?" [The first, in Chapter 1, was $y = x$. It's a special case of a direct variation $y = kx$, with $k = 1$. And direct variations are special cases of the intercept form $a + bx$, with $a = 0$ and $b = k$.]

Elicit the fact that the coefficient of x, no matter what it's called, gives the rate of change as well as the common difference between consecutive terms in the recursive sequence.

[Ask] "Which variables might be called input and output variables for the investigation?" [The input variable is the number of minutes spent exercising. The output variable is the number of calories burned.] See Lesson 7.3 for a discussion of independent and dependent variables.

Assessing Progress
Your observations should allow you to assess students' abilities with recursive routines, calculator lists, scatter plots, graphing equations, and undoing operations.

▸ *EXAMPLE A*

Besides revisiting the ideas of the investigation, this example shows how the graph of the linear equation $y = a + bx$ is a shift of the graph of $y = bx$. Translation will come up in Chapter 8.

Be sure students understand that the second column in the given table refers only to the calories burned while swimming, whereas the third column refers to the total number of calories burned, including those Sam burned before he started swimming.

Some students may say one graph is a vertical shift of the other. Other students, considering the graphs over quadrants other than the first, may see it as a horizontal shift. Encourage both viewpoints. Explore why they are equivalent and how the constants affect the amount of shift.

In the equation $y = 215 + 3.8x$, 215 is the value of a. It represents the 215 calories Manisha burned while jogging before her workout. The value of b is 3.8. It represents the rate her body burned calories while she was pedaling. What would happen if Manisha chose a different physical activity before pedaling on the stationary bike?

You can also think of direct variations in the form $y = kx$ as equations in intercept form. For instance, Sam's trainer tells him that swimming will burn 7.8 calories per minute. When the time spent swimming is 0, the number of calories burned is 0, so a is 0 and drops out of the equation. The number of calories burned is proportional to the time spent swimming, so you can write the equation $y = 7.8x$.

The constant of variation k is 7.8, the rate at which Sam's body burns calories while he is swimming. It plays the same role as b in $y = a + bx$.

EXAMPLE A

Suppose Sam has already burned 325 calories before he begins to swim for his workout. His swim will burn 7.8 calories per minute.

a. Create a table of values for the calories Sam will burn by swimming 60 minutes and the total calories he will burn after each minute of swimming.

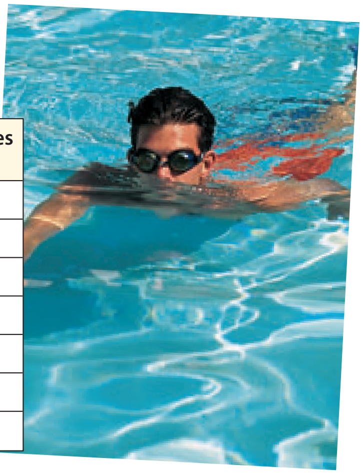
b. Define variables and write an equation in intercept form to describe this relationship.

c. On the same set of axes, graph the equation for total calories burned and the direct variation equation for calories burned by swimming.

d. How are the graphs similar? How are they different?

▸ *Solution*

a. The total numbers of calories burned appear in the third column of the table. Each entry is 325 plus the corresponding entry in the middle column.



Sam's Swim

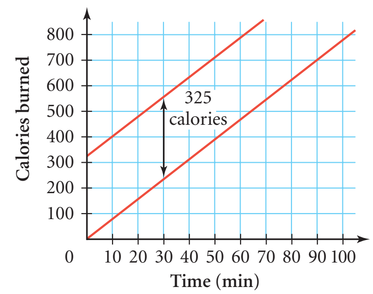
Swimming time (min)	Calories burned by swimming	Total calories burned
0	0	325
1	7.8	332.8
2	15.6	340.6
20	156	481
30	234	559
45	351	676
60	468	793

b. Let y represent the total number of calories burned, and let x represent the number of minutes Sam spends swimming.

$$y = 325 + 7.8x$$

c. The direct variation equation is $y = 7.8x$. Enter it into Y1 on your calculator. Enter the equation $y = 325 + 7.8x$ into Y2. Check to see that these equations give the same values as the table by looking at the calculator table.

d. The lower line shows the calories burned by swimming and is a direct variation. The upper line shows the total calories burned. It is 325 units above the first line because, at any particular time, Sam has burned 325 more calories. Both graphs have the same value of b, which is 7.8 calories per minute. The graphs are similar because both are lines with the same steepness. They are different because they have different y-intercepts.



What will different values of a in the equation $y = a + bx$ do to the graph?

EXAMPLE B

A minivan is 220 mi from its destination, Flint. It begins traveling toward Flint at 72 mi/h.

a. Define variables and write an equation in intercept form for this relationship.

b. Use your equation to calculate the location of the minivan after 2.5 h.

c. Use your equation to calculate when the minivan will be 130 mi from Flint.

d. Graph the relationship and locate the points that are the solutions to parts b and c.

e. What is the real-world meaning of the rate of change in this relationship? What does the sign of the rate of change indicate?

▶ Solution

a. Let the independent variable, x, represent the time in hours since the beginning of the trip. Let y represent the distance in miles between the minivan and Flint. The equation for the relationship is $y = 220 - 72x$.

b. Substitute the time, 2.5 h, for x.

$$y = 220 - 72 \cdot 2.5 = 40$$

So the minivan is 40 mi from Flint.

c. Substitute 130 mi for y and solve the equation $220 - 72x = 130$.

$220 + -72x = 130$	Original equation. The subtraction of $72x$ is written as addition of $-72x$.
$-72x = -90$	Subtract 220 to undo the addition.
$x = 1.25$	Divide by -72 to undo the multiplication.

The minivan will be 130 mi from Flint after 1.25 h. You can change 0.25 h to minutes using dimensional analysis. $0.25\text{ h} \cdot \frac{60\text{ min}}{1\text{ h}} = 15\text{ min}$, so you can also write the answer as 1 h 15 min.

▶ EXAMPLE B

Use this example if your students are having difficulty connecting graphs to equations or relating the mathematical theory to the real world.

In part a, students might use variables such as t for time. Often the text mirrors graphing calculators in using x for the independent variable. Emphasize dimensional analysis in setting up the equation.

Students might trace the graph of part d to answer part c.

Closing the Lesson

A linear equation in **intercept form,** $y = a + bx$, reflects the recursive routine used to generate a sequence of data values with a constant rate of change. Such a routine begins with a and adds b repeatedly. (The value of either a or b may be negative.)

BUILDING UNDERSTANDING

Students practice writing, graphing, and exploring linear equations, primarily in intercept form.

ASSIGNING HOMEWORK

Essential	**1, 2 or 3, 6, 7, 10**
Performance assessment	**9, 10**
Portfolio	**6**
Journal	**5, 7, 8**
Group	**7**
Review	**4, 5, 11–15**

Helping with the Exercises

Exercise 1 If students have difficulty relating recursive routines to the explicit equations, suggest that they make tables of data.

Exercises 2 and 3 These are the first equations in a while that don't use just x and y for variable names. Encourage students to write the dimensions of each number and variable, especially the rate.

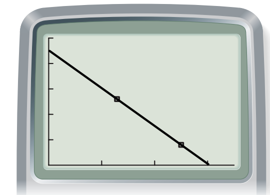
2c. 24 represents the initial number of miles the driver is from his or her destination.

2d. 45 means the driver is driving at a speed of 45 mi/h.

d. Set your calculator window to

[0, 3.5, 1, 0, 250, 50],

graph the equation, and press TRACE and the arrow keys to find the points where $x = 1.25$ and $x = 2.5$.



e. The rate of change indicates the speed of the car. If it is negative, the minivan is getting closer to Flint. That is, as time increases the distance decreases. A positive rate of change would mean that the vehicle was moving away from Flint.

In linear equations it is sometimes helpful to say which variable is the input variable and which is the output variable. The horizontal axis represents the input variable, and the vertical axis represents the output variable. In Example B, the input variable, x, represents time so the x-axis is labeled time, and the output variable, y, represents distance so the y-axis is labeled distance. What are the input and output variables in the investigation and in Example A?
The input variable is time, and the output variable is calories burned.

EXERCISES

You will need your graphing calculator for Exercises **2, 3, 6,** and **9.**

Practice Your Skills

1. Match the recursive routine in the first column with the equation in the second column.

ii **a.** 3 ENTER
Ans + 4 ENTER; ENTER, . . . @

i. $y = 4 - 3x$

iv **b.** 4 ENTER
Ans + 3 ENTER; ENTER, . . .

ii. $y = 3 + 4x$

iii **c.** -3 ENTER
Ans − 4 ENTER; ENTER, . . .

iii. $y = -3 - 4x$

i **d.** 4 ENTER
Ans − 3 ENTER; ENTER, . . .

iv. $y = 4 + 3x$

2. You can use the equation $d = 24 - 45t$ to model the distance from a destination for someone driving down the highway, where distance d is measured in miles and time t is measured in hours. Graph the equation and use the trace function to find the approximate time for each distance given in 2a and b.

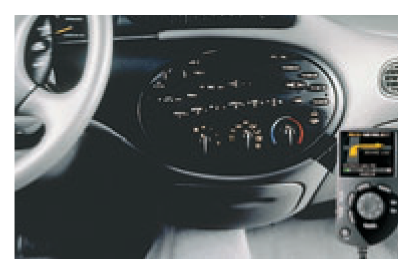
a. $d = 16$ mi @ $t \approx 0.18$ h

b. $d = 3$ mi $t \approx 0.47$ h

c. What is the real-world meaning of 24? @

d. What is the real-world meaning of 45?

e. Solve the equation $24 - 45t = 16$. $t = \frac{8}{45}$, or $0.1\overline{7}$



Some rental cars have in-dash navigation systems.

3. You can use the equation $d = 4.7 + 2.8t$ to model a walk in which the distance from a motion sensor d is measured in feet and the time t is measured in seconds. Graph the equation and use the trace function to find the approximate distance from a motion sensor for each time value given in 3a and b.

a. $t = 12$ s $d \approx 38.3$ ft

b. $t = 7.4$ s $d \approx 25.42$ ft

c. What is the real-world meaning of 4.7?
The walker started 4.7 ft away from the motion sensor.

d. What is the real-world meaning of 2.8?
The walker was walking at a rate of 2.8 ft/s.

4. Undo the order of operations to find the x-value in each equation.

a. $3(x - 5.2) + 7.8 = 14$ @ $x \approx 7.267$

b. $3.5\left(\frac{x-8}{4}\right) = 2.8$ $x = 11.2$

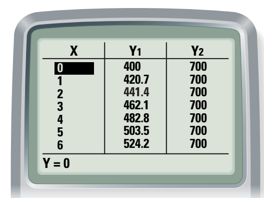
5. The equation $y = 35 + 0.8x$ gives the distance a sports car is from Flint after x minutes.

a. How far is the sports car from Flint after 25 minutes? $35 + 0.8(25) = 55$ mi

b. How long will it take until the sports car is 75 miles from Flint? Show how to find the solution using two different methods. 50 min; students might use a graph or the undo method.

Reason and Apply

6. **APPLICATION** Louis is beginning a new exercise workout. His trainer shows him the calculator table with x-values showing his workout time in minutes. The Y_1-values are the total calories Louis burned while running, and the Y_2-values are the number of calories he wants to burn.



a. Find how many calories Louis has burned before beginning to run, how many he burns per minute running, and the total calories he wants to burn. ⓗ

b. Write a recursive routine that will generate the values listed in Y_1. @ 400 ENTER, Ans + 20.7 ENTER

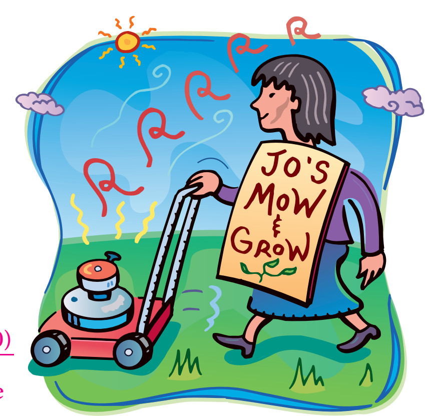
c. Use your recursive routine to write a linear equation in intercept form. Check that your equation generates the table values listed in Y_1. $Y_1 = 400 + 20.7x$

d. Write a recursive routine that will generate the values listed in Y_2. @ 700 ENTER, Ans + 0 ENTER

e. Write an equation that generates the table values listed in Y_2. @ $Y_2 = 700 + 0x$ or $Y_2 = 700$

f. Graph the equations in Y_1 and Y_2 on your calculator. Your window should show a time of up to 30 minutes. What is the real-world meaning of the y-intercept in Y_1?

g. Use the trace function to find the approximate coordinates of the point where the lines meet. What is the real-world meaning of this point?



7. Jo mows lawns after school. She finds that she can use the equation $P = -300 + 15N$ to calculate her profit.

a. Give some possible real-world meanings for the numbers -300 and 15 and the variable N.

b. Invent two questions related to this situation and then answer them.

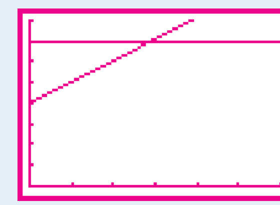
c. Solve the equation $P = -300 + 15N$ for the variable N. $N = \frac{(P + 300)}{15}$

d. What does the equation in 7c tell you? It tells you the number of lawns you have to mow to make a certain amount of profit.

Exercise 6 In Example B, students moved the cursor to trace a graph and find specific points on the graph. In this exercise, they trace to find the intersection of a horizontal line $y = 700$ with the line $y = 400 + 20.7x$.

6a. Louis has burned 400 calories before beginning to run. His calorie-burning rate is 20.7 calories per minute, and he wants to burn 700 total calories.

6f. The y-intercept of Y_1, which is 400, is the number of calories burned after 0 min of running (before Louis begins to run).



[0, 30, 5, 0, 800, 100]

6g. The approximate coordinates of the point where the lines meet are (14.5, 700). This means that after 14.5 min of running, Louis will have burned off his desired total of 700 calories.

Exercise 7 Students may be confused by the variable names. Suggest that they write the appropriate dimensions.

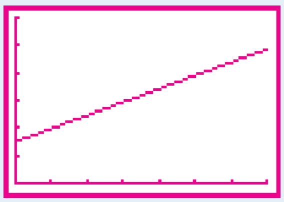
7a. One possible scenario: Jo has an initial start-up cost of \$300 for equipment and expenses. She makes \$15 for every lawn she mows, N.

7b. Sample questions: How many lawns will Jo have to mow to break even? [Solve the equation $-300 + 15N = 0$; Jo must mow 20 lawns.] How much profit will Jo earn if she mows 40 lawns? [Substitute 40 for N; \$300.]

Exercise 9 If students are having difficulty writing an equation, encourage them to generate the data with a recursive routine. As needed, point out that 12% is a rate.

9a. $y = 45 + 0.12x$, where x represents dollar amounts customers spend and y represents Manny's daily income in dollars

9b.



[0, 840, 120, 0, 180, 30]

Exercise 10 **[Alert]** In 10a–c, students might introduce several independent variables. For example, in 10a students might introduce b for minutes biked and s for minutes swum. They may or may not then replace all but one of these variables with the given constant(s)—in 10a, replacing b with 30. As needed, help students see that the question in 10d refers to each of the days described in 10a–c.

8. As part of a physics experiment, June threw an object off a cliff and measured how fast it was traveling downward. When the object left June's hand, it was traveling 5 m/s, and it sped up as it fell. The table shows a partial list of the data she collected as the object fell.

Time (s)	Speed (m/s)
0	5.0
0.5	9.9
1.0	14.8
1.5	19.7

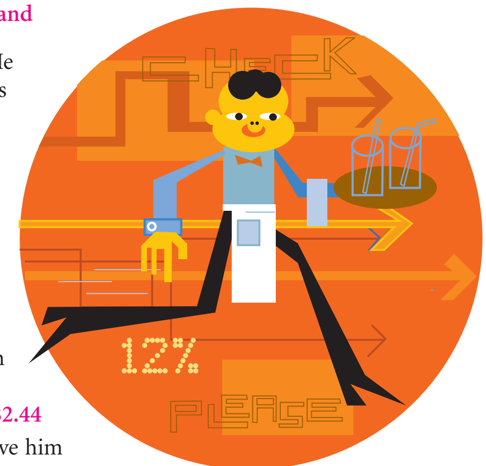
a. Write an equation to represent the speed of the object. Ⓐ $s = 5 + 9.8t$ or $s = 9.8t + 5$

b. What was the object's speed after 3 s? 34.4 m/s

c. If it were possible for the object to fall long enough, how many seconds would pass before it reached a speed of 83.4 m/s? Ⓐ 8 s

d. What limitations do you think this equation has in modeling this situation? Ⓐ It doesn't account for air resistance and terminal speed.

9. **APPLICATION** Manny has a part-time job as a waiter. He makes \$45 per day plus tips. He has calculated that his average tip is 12% of the total amount his customers spend on food and beverages.



a. Define variables and write an equation in intercept form to represent Manny's daily income in terms of the amount his customers spend on food and beverages. Ⓗ

b. Graph this relationship for food and beverage amounts between \$0 and \$900.

c. Write and evaluate an expression to find how much Manny makes in one day if his customers spend \$312 on food and beverages. $45 + 0.12 \cdot 312 = \$82.44$

d. What amounts spent on food and beverages will give him a daily income between \$105 and \$120? between \$500 and \$625

10. **APPLICATION** Paula is cross-training for a triathlon in which she cycles, swims, and runs. Before designing an exercise program for Paula, her coach consults a table listing rates for calories burned in various activities.

Cross-training activity	Calories burned (per min)
Walking	3.2
Bicycling	3.8
Swimming	6.9
Jogging	7.3
Running	11.3

a. On Monday, Paula starts her workout by biking for 30 minutes and then swimming. Write an equation for the calories she burns on Monday in terms of the number of minutes she swims. $y = 114 + 6.9x$

b. On Wednesday, Paula starts her workout by swimming for 30 minutes and then jogging. Write an equation for the number of calories she burns on Wednesday in terms of the number of minutes she jogs. $y = 207 + 7.3x$

c. On Friday, Paula starts her workout by swimming 15 minutes, then biking for 15 minutes, then running. Write an equation for the number of calories she burns on Friday in terms of the number of minutes she spends running. $y = 160.5 + 11.3x$

d. How many total calories does Paula burn on each day described in 10a–c if she does a 60-minute workout? Monday: 321 calories; Wednesday: 426 calories; Friday: 499.5 calories

Review

2.2 **11.** At a family picnic, your cousin tells you that he always has a hard time remembering how to compute percents. Write him a note explaining what percent means. Use these problems as examples of how to solve the different types of percent problems, with an answer for each.

a. 8 is 15% of what number? @ $\frac{8}{n} = \frac{15}{100}$, $n \approx 53.3$

b. 15% of 18.95 is what number? $\frac{15}{100} = \frac{n}{18.95}$, $n \approx 2.8$

c. What percent of 64 is 326? $\frac{p}{100} = \frac{326}{64}$, $p \approx 509.4$

d. 10% of what number is 40? $\frac{10}{100} = \frac{40}{n}$, $n = 400$

Exercise 11 Students can complete this exercise using proportions as they learned in Chapter 2, but allow other correct methods.

11. Partial answer: Write the percent as one ratio of a proportion. Put the part over the whole in the other ratio.

2.3 **12.** **APPLICATION** Carl has been keeping a record of his gas purchases for his new car. Each time he buys gas, he fills the tank completely. Then he records the number of gallons he bought and the miles since the last fill-up. Here is his record:

Carl's Purchases

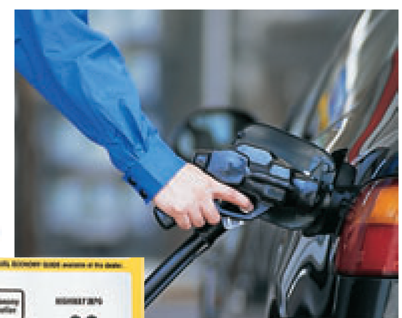
Miles traveled	Gallons	$\frac{\text{miles}}{\text{gallon}}$
363	16.2	22.4
342	15.1	22.6
285	12.9	22.1

a. Copy and complete the table by calculating the ratio of miles per gallon for each purchase.

b. What is the average rate of miles per gallon so far? 22.4 mi/gal

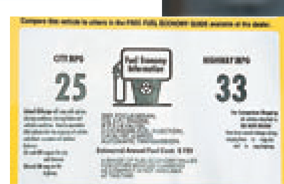
c. The car's tank holds 17.1 gallons. To the nearest mile, how far should Carl be able to go without running out of gas? 383 mi

d. Carl is planning a trip across the United States. He estimates that the trip will be 4230 miles. How many gallons of gas can Carl expect to buy? approximately 189 gal



Consumer CONNECTION

Many factors influence the rate at which cars use gas, including size, age, and driving conditions. Advertisements for new cars often give the average mpg for city traffic (slow, congested) and highway traffic (fast, free flowing). These rates help consumers make an informed purchase. For more information about fuel economy, see the links at www.keymath.com/DA .



3.2 **13.** Match each recursive routine to a graph below. Explain how you made your decision and tell what assumptions you made.

ii **a.** 2.5 ENTER
Ans + 0.5 ENTER; ENTER, . . .

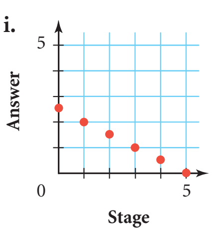
iv **b.** 1.0 ENTER
Ans + 1.0 ENTER; ENTER, . . .

iii **c.** 2.0 ENTER
Ans + 1.0 ENTER; ENTER, . . .

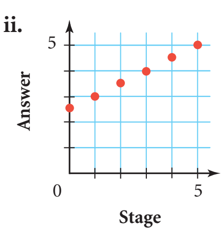
i **d.** 2.5 ENTER
Ans − 0.5 ENTER; ENTER, . . .

13. Sample explanation: I matched the rate of change to each graph. I assumed the starting value was the y-intercept.

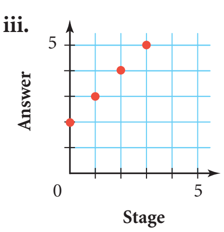
i.



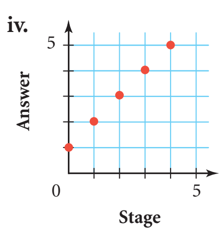
ii.



iii.



iv.



14c. {0,0} ENTER, {Ans(1) + 1, Ans(2) + 14} ENTER

14d. The points lie on a line.

3.3 **14.** Bjarne is training for a bicycle race by riding on a stationary bicycle with a time-distance readout. He is riding at a constant speed. The graph shows his accumulated distance and time as he rides.

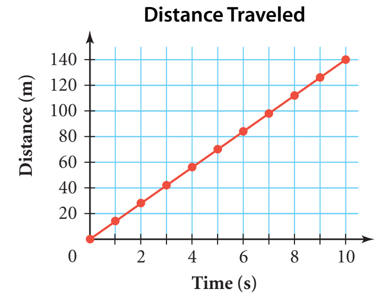
a. How fast is Bjarne bicycling? 14 m/s

b. Copy and complete the table. @

c. Write a recursive routine for Bjarne's ride.

d. Looking at the graph, how do you know that Bjarne is neither slowing down nor speeding up during his ride?

e. If Bjarne keeps up the same pace, how far will he ride in one hour? 50,400 m, or 50.4 km



Bicyclists race through San Luis Obispo, California.

Time (s)	Distance (m)
1	14
2	28
3	42
4	56
5	70
6	84
7	98
8	112
9	126
10	140

15a. The expression equals -4.

Ans − 8	-3
Ans · 4	-12
Ans/3	-4

2.8 **15.** Consider the expression $\frac{4(y-8)}{3}$.

a. Find the value of the expression if $y = 5$. Make a table to show the order of operations. @

b. Solve the equation $\frac{4(y-8)}{3} = 8$ by undoing the sequence of operations. @ $y = 14$

IMPROVING YOUR REASONING SKILLS

You have two containers of the same size; one contains juice and the other contains water. Remove one tablespoon of juice and put it into the water and stir. Then remove one tablespoon of the water and juice mixture and put it into the juice. Is there more water in the juice or more juice in the water?

IMPROVING REASONING SKILLS

If students are having difficulty figuring out that the percentage of juice in the water is the same as the percentage of the water in the juice, you might suggest that they think about particular amounts of liquid, such as 10 oz of each with 1 oz being transferred. Or produce some decks of playing cards. Give each pair of students a pile of ten red cards and a pile of ten black cards. Have one student pull any number of red cards out and mix them among the black cards. The other student pulls out the same number of cards from the other pile and puts them into the red pile. Keeping track of how many of each color are moving and trying it with extreme cases might deepen students' understanding of the ratios involved.

LESSON

3.5

Linear Equations and Rate of Change

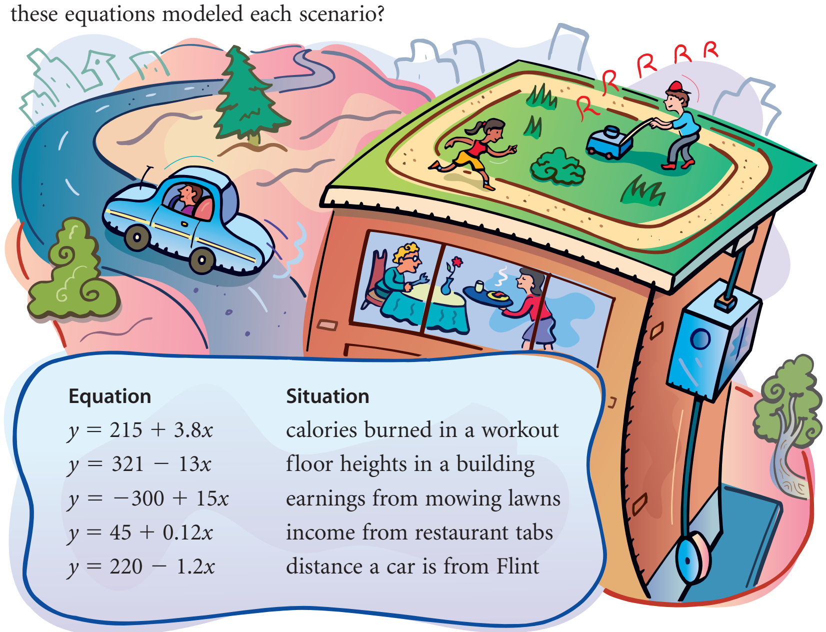
How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?

ALBERT EINSTEIN

In this lesson you will continue to develop your skills with equations, graphs, and tables of data by exploring the role that the value of b plays in the equation

$$y = a + bx$$

You have already studied the intercept form of a linear equation in several real-world situations. You have used the intercept form to relate calories to minutes spent exercising, floor heights to floor numbers, and distances to time. So, defining variables is an important part of writing equations. Depending on the context of an equation, its numbers take on different real-world meanings. Can you recall how these equations modeled each scenario?



Equation	Situation
$y = 215 + 3.8x$	calories burned in a workout
$y = 321 - 13x$	floor heights in a building
$y = -300 + 15x$	earnings from mowing lawns
$y = 45 + 0.12x$	income from restaurant tabs
$y = 220 - 1.2x$	distance a car is from Flint

Winds of 40 mi/h blow on North Michigan Ave. in 1955 Chicago.

In most linear equations, there are different output values for different input values. This happens when the coefficient of x is not zero. You'll explore how this coefficient relates input and output values in the examples and the investigation.

In addition to giving the actual temperature, weather reports often indicate the temperature you *feel* as a result of the wind chill factor. The wind makes it feel colder than it actually is. In the next example you will use recursive routines to answer some questions about wind chill.

LESSON

3.5

PLANNING

LESSON OUTLINE

One day:

5 min	Introduction
5 min	Example A
20 min	Investigation
5 min	Sharing
5 min	Example B
5 min	Closing
5 min	Exercises

MATERIALS

- Wind Chill (T), *optional*
- Calculator Note 3C
- Sketchpad demonstration Rate of Change, *optional*
- CBL 2 demonstration Heating Up, *optional*

TEACHING

[ELL] Input variables are those that are put in. Output variables are those that come out.

Students get closer to the notion of slope by studying rates of change as they construct linear equations and their graphs from input-output tables. The concept of input and output variables is a precursor to the study of functions. (The input value is the *independent variable* and the output value is the *dependent variable,* although those terms aren't used until Lesson 7.3.)

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number		Problem Solving
✓	Algebra	✓	Reasoning
✓	Geometry	✓	Communication
✓	Measurement	✓	Connections
	Data/Probability	✓	Representation

LESSON OBJECTIVES

- Interpret equations in intercept form using input and output variables
- Explore the relationships among tables, scatter plots, recursive routines, and equations

▶ ***EXAMPLE A***

This example introduces the notion of an input-output table and revisits the idea of rate of change. Students may be confused by the use of table entries as units rather than degrees. That is, the rate of change is 6.4° wind chill per table entry, or 1.28° wind chill per degree temperature.

Wind chill for other wind speeds is given in the investigation (20 mi/h), Exercise 2 (40 mi/h), and Exercise 6 (35 mi/h). The faster the wind, the greater the wind chill factor. Although wind chill appears to be linearly related to the wind speeds, it levels off at wind speeds greater than 45 mi/h.

Guiding the Investigation

One Step
Show the Wind Chill transparency or refer to the table at the bottom of this page. **[Ask]** "What is the actual temperature if the weather report indicates a wind chill of 10.6° at this wind speed?" As students work, encourage them to use recursive routines, calculate rates of change, and use equations and graphs.

Step 1 In real life, it's not always clear which variable denotes input and which denotes output. The convention is to put input values in the left column and output values in the right column.

EXAMPLE A

The table relates the approximate wind chills for different actual temperatures when the wind speed is 15 mi/h. Assume the wind chill is a linear relationship for temperatures between −5° and 35°.

Temperature (°F)	−5	0	5	10	15	20	25	30	35
Wind chill (°F)	−25.8	−19.4	−13			6.2		19	25.4

a. What are the input and output variables?

b. What is the change in temperature from one table entry to the next? What is the corresponding change in the wind chill?

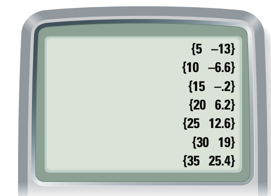
c. Use calculator lists to write a recursive routine that generates the table values. What are the missing entries?

▶ ***Solution***

a. The input variable is the actual air temperature in °F. The output variable is the temperature you feel as a result of the wind chill factor.

b. For every 5° increase in temperature, the wind chill increases 6.4°.

c. The recursive routine to complete the missing table values is {−5, −25.8} ENTER and {Ans(1) + 5, Ans(2) + 6.4} ENTER. The calculator screen displays the missing entries.



In Example A, the *rate* at which the wind chill drops can be calculated from the ratio $\frac{6.4}{5}$, or $\frac{1.28}{1}$. In other words, it feels 1.28° colder for every 1° drop in air temperature. This number is the rate of change for a wind speed of 15 mi/h. The **rate of change** is equal to the ratio of the change in output values divided by the corresponding change in input values.

Do you think the rate of change differs with various wind speeds?

Investigation
Wind Chill

In this investigation you'll use the relationship between temperature and wind chill to explore the concept of rate of change and its connections to tables, scatter plots, recursive routines, equations, and graphs.

The data in the table represent the approximate wind chill temperatures in degrees Fahrenheit for a wind speed of 20 mi/h. Use this data set to complete each task.

Temperature (°F)	Wind chill (°F)
−5	−28.540
0	−21.980
1	−20.668
2	−19.356
5	−15.420
15	−2.300
35	23.940

[Data sets: **TMPWS, WNDCH**]

Step 1 Let x be the input variable representing the temperature in °F, and let y be the output variable representing wind chill in °F.

Step 1 Define the input and output variables for this relationship.

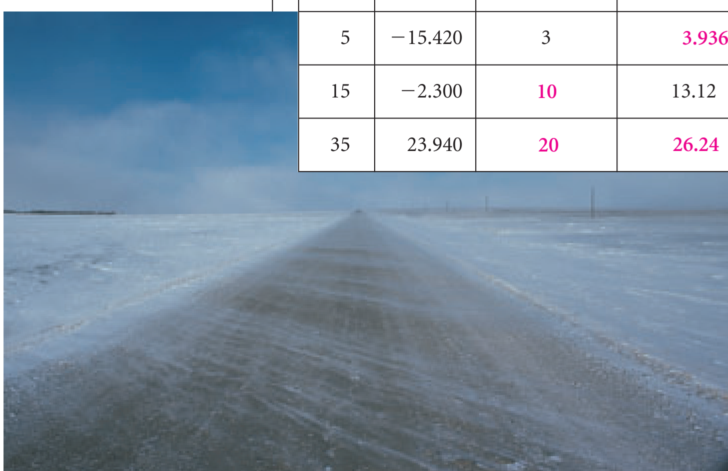
Step 3 Starting value: $\{-5, -28.54\}$ ENTER
Rule for 1° increment: $\{\text{Ans}(1) + 1, \text{Ans}(2) + 1.312\}$

Step 2 Plot the points and describe the viewing window you used.

Step 3 Write a recursive routine that gives the pairs of values listed in the table.

Step 4 Copy the table. Complete the third and fourth columns of the table by recording the changes between consecutive input and output values. Then find the rate of change.

Input	Output	Change in input values	Change in output values	Rate of change
−5	−28.540			
0	−21.980	5	6.56	$\frac{+6.56}{+5} =$ 1.312
1	−20.668	1	1.312	$\frac{+1.312}{+1} =$ 1.312
2	−19.356	1	1.312	$\frac{+1.312}{+1} =$ 1.312
5	−15.420	3	3.936	$\frac{+3.936}{+3} =$ 1.312
15	−2.300	10	13.12	$\frac{+13.12}{+10} =$ 1.312
35	23.940	20	26.24	$\frac{+26.24}{+20} =$ 1.312



High wind speeds in Saskatchewan, Canada, drop temperatures below freezing.

Step 5 $y = -21.980 + 1.312x$; the rule appears as the coefficient of x.

Step 5 Use your routine to write a linear equation in intercept form that relates wind chill to temperature. Note that the starting value, −28.540, is not the y-intercept. How does the rule of the routine appear in your equation?

Step 6 Graph the equation on the same set of axes as your scatter plot. Use the calculator table to check that your equation is correct. Does it make sense to draw a line through the points? Where does the y-intercept show up in your equation?

Step 7 What do you notice about the values for rate of change listed in your table? How does the rate of change show up in your equation? In your graph?

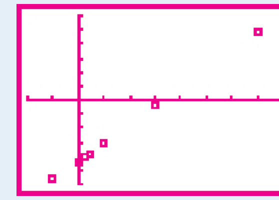
Step 8 Explain how to use the rate of change to find the actual temperature if the weather report indicates a wind chill of 9.5° with 20 mi/h winds.

SHARING IDEAS

Have students present several approaches to Step 8, describing their equations in the process. Mention that the constant rate of change is sometimes called the *wind chill factor* at this wind speed. Draw out the fact that a rate of change is a rate as defined in Chapter 2—that is, a ratio with denominator 1. The rate of change is the output change with each additional unit of input.

[Ask] "What do the equations have in common?" [Elicit the idea that the output variable is usually isolated on the left side, making it easy to enter functions in the calculator for graphing. The right side is like the recursive routine: A constant corresponds to the starting value, and the rate of change is multiplied by the input variable.]

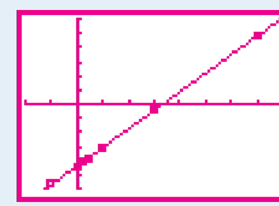
Step 2



$[-10, 40, 5, -30, 30, 5]$

Step 5 If students are having difficulty, remind them that they can find the rate of change easily if they know output values for input values that are 1 unit apart. Also, elicit the idea that an input value of 0 gives the y-intercept as an output value.

Step 6 Yes, a line represents every possible temperature. The y-intercept shows up as the value of a in the equation. It is not the starting value of the routine.



Step 7 The values for rate of change are all equivalent. The rate of change appears as b, the coefficient of x. In the graph, to go from one point to the next you move right 1 unit and up 1.312 units, which is the rate of change.

Step 8 Encourage a variety of approaches: tracing, calculator tables, or working backward to solve the equation $9.5 = -21.98 + 1.312x$.

Step 8 Explanations will vary. Students can add 1.312° nine times to −2.3 and add 9° to 15°. They can also subtract 1.312° eleven times from 23.940° and subtract 11° from 35°. The answer is 24°F.

[Ask] "How is the rate of change represented on the graph?" As needed, draw a picture showing that, as the line moves 1 unit from left to right, it rises by the amount of the rate of change, so the larger the rate of change, the steeper the line. You might use the word *slope* as a synonym for steepness of the line, representing the rate of change. The term *slope* will be defined in Lesson 4.1.

For more practice with linear equations, you might use the CBL 2 demonstration Heating Up, which has students model the relationship between Fahrenheit and Celsius.

Assessing Progress
Observe students' skills at making a scatter plot, choosing a viewing window, writing a recursive routine, writing a linear equation from a recursive routine, and graphing a line. Also assess their understanding of rate of change.

▸ EXAMPLE B

To encourage critical thinking, ask students if they believe the given data. They may speculate that the actual numbers depend on the time of year and the location. Part c gives you the chance to connect the mathematics to the real-world context.

EXAMPLE B

This table shows the temperature of the air outside an airplane at different altitudes.

Input	Output
Altitude (m)	Temperature (°C)
1000	7.7
1500	4.2
2200	−0.7
3000	−6.3
4700	−18.2
6000	−27.3

a. Add three columns to the table, and record the change in input values, the change in output values, and the corresponding rate of change.

b. Use the table and a recursive routine to write a linear equation in intercept form $y = a + bx$.

c. What are the real-world meanings of the values for a and b in your equation?

▸ Solution

a. Record the change in input values, change in output values, and rate of change in a table. Note the units of each value.

Input	Output			
Altitude (m)	Temperature (°C)	Change in input values (m)	Change in output values (°C)	Rate of change (°C/m)
1000	7.7			
1500	4.2	500	−3.5	$\frac{-3.5}{500} = -0.007$
2200	−0.7	700	−4.9	$\frac{-4.9}{700} = -0.007$
3000	−6.3	800	−5.6	$\frac{-5.6}{800} = -0.007$
4700	−18.2	1700	−11.9	$\frac{-11.9}{1700} = -0.007$
6000	−27.3	1300	−9.1	$\frac{-9.1}{1300} = -0.007$

b. Note that the rate of change, or slope, is always -0.007, or $\frac{-7}{1000}$. You can also write the rate of change as $\frac{-0.7}{100}$, so this recursive routine models the relationship:

{1000, 7.7} ENTER

{Ans(1) + 100, Ans(2) − 0.7} ENTER

Working this routine backward, {Ans(1) − 100, Ans(2) + 0.7}, will eventually give the result {0, 14.7}. So the intercept form of the equation is $y = 14.7 - 0.007x$, where x represents the altitude in meters and y represents the air temperature in °C.

Note that the starting value of the recursive routine is not the same as the value of the y-intercept in the equation.

c. The value of a, 14.7, is the temperature (in °C) of the air at sea level. The value of b indicates that the temperature drops 0.007°C for each meter that a plane climbs.

EXERCISES

You will need your graphing calculator for Exercises **4, 5,** and **10.**

Practice Your Skills

1. Copy and complete the table of output values for each equation.

a. $y = 50 + 2.5x$ @

Input x	Output y
20	100
−30	−25
16	90
15	87.5
−12.5	18.75

b. $L_2 = -5.2 - 10 \cdot L_1$

L_1 x	L_2 y
0	−5.2
−8	74.8
24	−245.2
−35	344.8
−5.2	46.8

2. Use the equation $w = -29 + 1.4t$, where t is temperature and w is wind chill, both in °F, to approximate the wind chill temperatures for a wind speed of 40 mi/h.

a. Find w for $t = 32°$. $w = 15.8$°F

b. Find t for a wind chill of $w = -8$. @ $w = 15$°F

c. What is the real-world meaning of 1.4? @

d. What is the real-world meaning of −29?

3. Describe what the rate of change looks like in each graph.

a. the graph of a person walking at a steady rate toward a motion sensor @

b. the graph of a person standing still

c. the graph of a person walking at a steady rate away from a motion sensor

d. the graph of one person walking at a steady rate faster than another person

4. Use the "Easy" setting of the INOUT game on your calculator to produce four data tables. Copy each table and write the equation you used to match the data values in the table. [▶ See **Calculator Note 3C** to learn how to run the program. ◀]

3c. The rate is positive, so the line goes from the lower left to the upper right.

3d. The rate for the speedier walker will be greater than the rate for the person walking more slowly, so the graph for the speedier walker will be steeper than the graph for the slower walker.

4. A sample:

```
 IN OUT
          [[0  -5]
           [1  -3]
           [2  -1]
           [3  1 ]
           [4  3 ]
           [5  5 ]]
GUESS: -5+2L1
```

Closing the Lesson

You can calculate the **rate of change** as a difference of output values divided by a difference of corresponding input values. The rate of change determines the steepness of the graph of the linear equation representing the data.

BUILDING UNDERSTANDING

Students work more with input and output variables, rates of change, and graphs of equations.

ASSIGNING HOMEWORK

Essential	1–3, 6–8
Performance assessment	5, 6, 7
Portfolio	10
Journal	2, 8
Group	4, 5, 7, 9
Review	11–14

Helping with the Exercises

Exercise 2 If students are having difficulty, encourage them to measure or count how many units the line rises or falls as it moves 1 unit from left to right.

2c. The wind chill temperature changes by 1.4° for each 1° change in actual temperature.

2d. If the actual temperature is 0°F, the wind chill temperature is −29°F.

3a. The rate is negative, so the line goes from the upper left to the lower right.

3b. The rate is neither negative nor positive but zero. The line is a horizontal line.

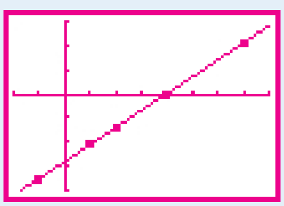
Exercise 6 **[Language]** Students may wonder how to answer the question in 6d about the difference in the graphs. Encourage them to use the words *discrete* and *continuous.*

The Sketchpad demonstration Rate of Change can replace this exercise.

6a. The input variable x is the temperature in °F, and the output variable y is the wind chill in °F.

6b. The rate of change is 1.4°. For every 10° increase in temperature, there is a 14° increase in wind chill.

6d. Both graphs show linear relationships with identical rates of change and y-intercepts. The graphs are different in that the points are discrete and the equation continuous.



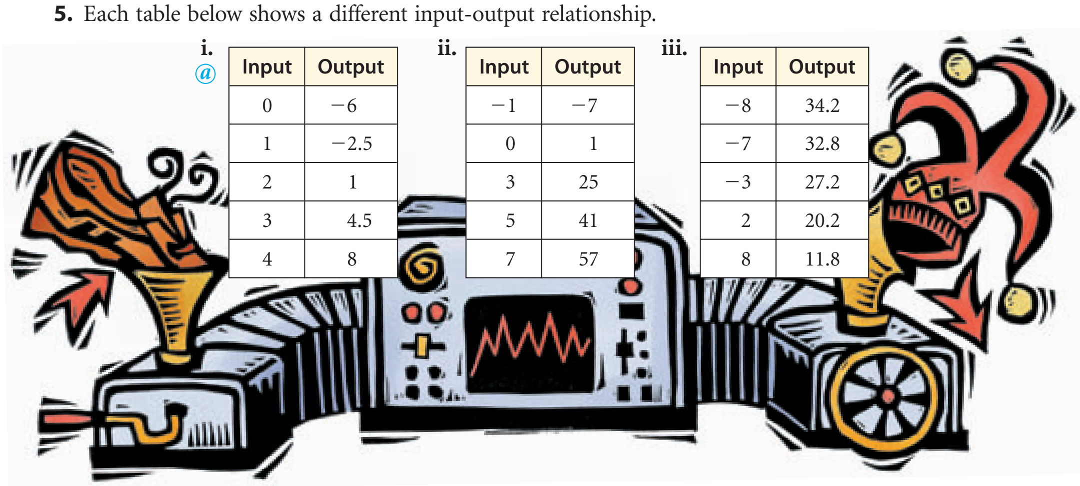
$[-10, 40, 5, -40, 30, 10]$

7a. *distance from sensor* $= 3.5 - 0.25 \cdot$ *time*

7b. 14 s after she begins walking

Reason and Apply

5. Each table below shows a different input-output relationship.



i. ⓐ

Input	Output
0	-6
1	-2.5
2	1
3	4.5
4	8

ii.

Input	Output
-1	-7
0	1
3	25
5	41
7	57

iii.

Input	Output
-8	34.2
-7	32.8
-3	27.2
2	20.2
8	11.8

a. Find the rate of change in each table. Explain how you found this value. **i.** 3.5; **ii.** 8; **iii.** -1.4

b. For each table, find the output value that corresponds to an input value of 0. What is this value called? ⓗ **i.** -6; **ii.** 1; **iii.** 23; the y-intercept

c. Use your results from 5a and b to write an equation in intercept form for each table. **i.** $y = -6 + 3.5x$; **ii.** $y = 1 + 8x$; **iii.** $y = 23 - 1.4x$

d. Use a calculator list of input values to check that each equation actually produces the output values shown in the table.

6. The wind chill temperatures for a wind speed of 35 mi/h are given in the table.

Temperature (°F)	-5	5	10	20	35
Wind chill (°F)	-35	-21	-14	0	21

a. Define input and output variables. ⓐ

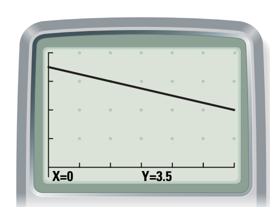
b. Find the rate of change. Explain how you got your answer. ⓐ

c. Write an equation in intercept form. ⓐ $y = -28 + 1.4x$

d. Plot the points and graph the equation on the same set of axes. How are the graphs for the points and the equation similar? How are they different?

7. Semantha's walk was recorded by a motion sensor. A graph of her walk and a few data points are shown here.

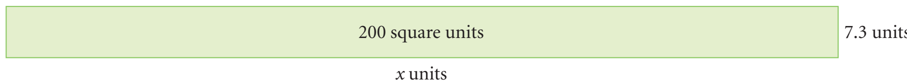
Time (s)	Distance (m)
0	3.5
2	3
6	2



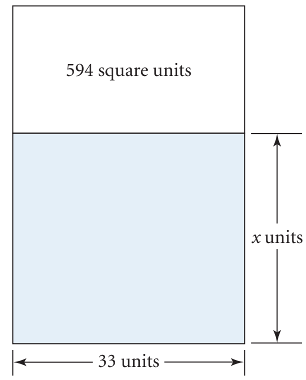
a. Write an equation in the form *Distance from sensor* = *start distance* + *change* to model this walk. ⓗ

b. If she continues to walk at a constant rate, at what time would she pass the sensor?

8. You can use the equation $7.3x = 200$ to describe a rectangle with an area of 200 square units like the one shown. What are the real-world meanings of the numbers and the variable in the equation? Solve the equation for x and explain the meaning of your solution. Is the rectangle drawn to scale? How can you tell?



9. The total area of the figure at right is 1584 square units. You can use the equation $1584 - 33x = 594$ to represent an area of 1584 square units minus the area of $33x$ square units. The area remaining is 594 square units.



a. What is the area of the shaded rectangle? @ 990 square units

b. Write the equation you would use to find the height of the shaded rectangle. @ possible answers: $33x = 990$; $x = \frac{990}{33}$

c. Solve the equation you wrote in 9b to find the height of the shaded rectangle. @ 30 units

10. Use the "Medium" setting of the INOUT game on your calculator to produce four data tables. Copy each table and write the equation you used to match the data values in the table. [▶ See Calculator Note 3C. ◀]

Review

2.8 **11.** Show how you can solve these equations by using an undoing process. Check your results by substituting the solutions into the original equations.

a. $-15 = -52 + 1.6x$

b. $7 - 3x = 52$

3.2 **12.** **APPLICATION** To plan a trip downtown, you compare the costs of three different parking lots. ABC Parking charges \$5 for the first hour and \$2 for each additional hour or fraction of an hour. Cozy Car charges \$3 per hour or fraction of an hour, and The Corner Lot charges a \$15 flat rate for a whole day.

a. Make a table similar to the one shown. Write recursive routines to calculate the cost of parking up to 10 hours at each of the three lots.

Hours parked	ABC Parking	Cozy Car	The Corner Lot
1	5	3	15
2	7	6	15
3	9	9	15

b. Make three different scatter plots on the same pair of axes showing the parking rates at the three different lots. Use a different color for each parking lot. Put the hours on the horizontal axis and the cost on the vertical axis.

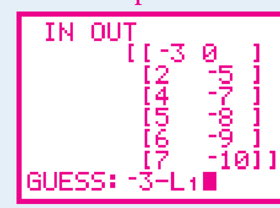
c. Compare the three scatter plots. Under what conditions is each parking lot the best deal for your trip? Use the graph to explain.

d. Would it make sense to draw a line through each set of points? Explain why or why not. No; because you have to pay for a whole hour for any fraction of the hour, the price of parking does not increase continuously.

Exercise 8 This exercise gives an opportunity to emphasize that a direct variation is a special kind of linear equation.

8. Because height times width gives area, 7.3 and x represent the height and width, respectively. The number 200 represents the area of the rectangle in square units. The solution is about 27.4 units. The rectangle is not drawn to scale. The length should be about 3.8 times the width.

10. A sample:



Exercise 12b On calculators without a color display, students can use different graph marks.

12a.

Hours	ABC	Cozy	Corner
4	11	12	15
5	13	15	15
6	15	18	15
7	17	21	15
8	19	24	15
9	21	27	15
10	23	30	15

ABC: {1,5},
{Ans(1) + 1, Ans(2) + 2};
Cozy: {1,3},
{Ans(1) + 1, Ans(2) + 3};
Corner: {1,15},
{Ans(1) + 1, Ans(2) + 0}

12c. For less than 3 h, Cozy Car is the least expensive option because on the graph its points are lower than the points of the others. For exactly 3 h, ABC and Cozy Car cost the same. For 3 to 5 h, ABC Parking has the best price, because its graph has a lower cost in that time frame. For exactly 6 h, ABC and The Corner Lot cost the same. For more than 6 h, The Corner Lot is the least expensive option.

11a.

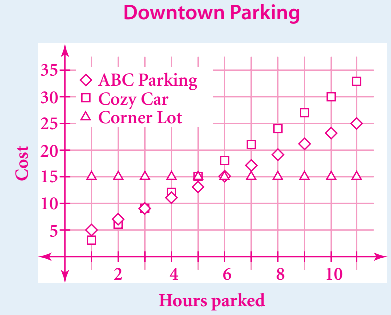
-15	-15
Ans + 52	37
Ans/1.6	23.125
$-52 + 1.6(23.125) = -15$	Check.

11b.

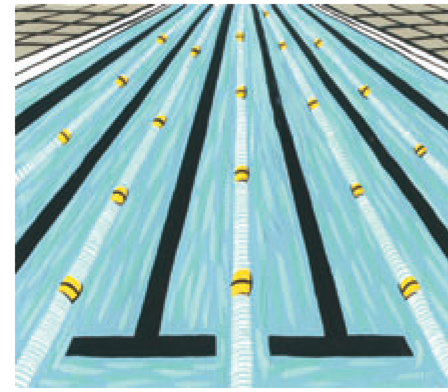
52	52
Ans − 7	45
Ans/−3	-15
$7 - 3(-15) = 52$	Check.

12b.

Downtown Parking



2.3 **13.** Today while Don was swimming, he started wondering how many lengths he would have to swim in order to swim different distances. At one end of the pool, he stopped, gasping for breath, and asked the lifeguard. She told him that 1 length of the pool is 25 yards and that 72 lengths is 1 mile. As he continued swimming, he wondered:



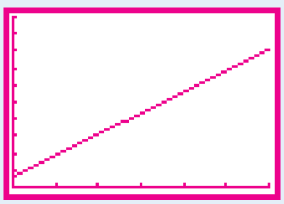
70.4 lengths **a.** Is 72 lengths really a mile? Exactly how many lengths would it take to swim a mile? ⓗ

b. If it took him a total of 40 minutes to swim a mile, what was his average speed in feet per second? 2.2 ft/s

c. How many lengths would it take to swim a kilometer? about 44 lengths

d. Last summer Don got to swim in a pool that was 25 meters long. How many lengths would it take to swim a kilometer there? How many for a mile? 40 lengths for a kilometer; about 64 lengths for a mile

14b.



[0, 60, 10, 0, 100, 10]

3.4 **14.** **APPLICATION** Holly has joined a video rental club. After paying \$6 a year to join, she then has to pay only \$1.25 for each new release she rents.

a. Write an equation in intercept form to represent Holly's cost for movie rentals. ⓐ $y = 6 + 1.25x$

b. Graph this situation for up to 60 movie rentals.

c. Video Unlimited charges \$60 for a year of unlimited movie rentals. How many movies would Holly have to rent for this to be a better deal? 44 movies

project

LEGAL LIMITS

To make a highway accessible to more vehicles, engineers reduce its steepness, also called its **gradient** or grade. This highway was designed with switchbacks so the gradient would be small.

A gradient is the inclination of a roadway to the horizontal surface. Research the federal, state, and local standards for the allowable gradients of highways, streets, and railway routes.

Find out how gradients are expressed in engineering terms. Give the standards for roadway types designed for vehicles of various weights, speeds, and engine power in terms of rate of change. Describe the alternatives available to engineers to reduce the gradient of a route in hilly or mountainous terrain. What safety measures do they incorporate to minimize risk on steep grades? Bring pictures to illustrate a presentation about your research, showing how engineers have applied standards to roads and routes in your home area.

Supporting the project

MOTIVATION

Steep roads are harder to travel and harder to maintain. Engineers must consider steepness as they design roads. What guidelines and regulations do they follow?

OUTCOMES

- ▸ Gradient is defined. Stated as a percent, 20% means a rise of 20 ft every 100 ft. Stated as a ratio, 1 to 5 is the same as 20%.
- ▸ The report summarizes standards. A 20% gradient, an angle of 11.5°, is considered steep. A maximum of 15% sustained gradient is recommended. Where there is heavy snowfall, the maximum is 10%.
- ▸ The report includes techniques for reducing gradients and safety measures that can minimize risk on steep grades.
- ▸ The report includes relevant pictures of nearby roads.
- • Gradients for roads are compared with gradients for railroads.
- • The slope of a line is carefully defined.

LESSON

3.6

Solving Equations Using the Balancing Method

Thinking in words slows you down and actually decreases comprehension in much the same way as walking a tightrope too slowly makes one lose one's balance.

LENORE FLEISCHER

In the previous two lessons, you learned about rate of change and the intercept form of a linear equation. In this lesson you'll learn symbolic methods to solve these equations. You've already seen the calculator methods of tracing on a graph and zooming in on a table. These methods usually give approximate solutions. Working backward to undo operations is a symbolic method that gives exact solutions. Another symbolic method that you can apply to solve equations is the **balancing method.** In this lesson you'll investigate how to use the balancing method to solve linear equations. You'll discover that it's closely related to the undoing method.

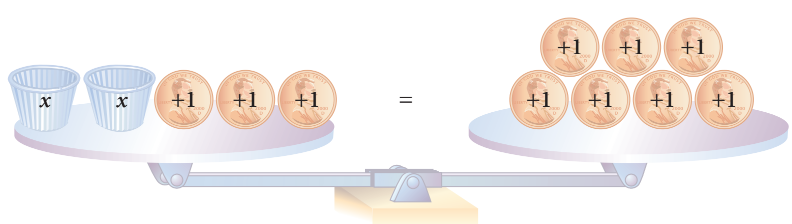
Investigation

Balancing Pennies

You will need

- pennies
- three paper cups

Here is a visual model of the equation $2x + 3 = 7$. A cup represents the variable x and pennies represent numbers. Assume that each cup has the same number of pennies in it and that the containers themselves are weightless.



Step 1 2; sample explanation: Four pennies must be in the two cups, so there are two per cup.

Step 1 How many pennies must be in each cup if the left side of the scale balances with the right side? Explain how you got your answer.

Your answer to Step 1 is the solution to the equation $2x + 3 = 7$. It's the number that can replace x to make the statement true. In Steps 2 and 3, you'll use pictures and equations to show stages that lead to the solution.

Step 2 $2x = 4$

Step 2 Redraw the picture above, but with three pennies removed from each side of the scale. Write the equation that your picture represents.

LESSON

3.6

PLANNING

LESSON OUTLINE

One day:

25 min	Investigation
5 min	Sharing
5 min	Example A
5 min	Example B
5 min	Closing
5 min	Exercises

MATERIALS

- pennies (different amounts from 20 to 60 per group) or squares of construction paper or bingo chips or algebra tiles
- 10 washers, *optional*
- paper cups (three per group)
- Pan Balance (T), *optional*
- Calculator Notes 2A, 3D
- Sketchpad demonstration The Balancing Method, *optional*

TEACHING

You can solve equations in which the unknown variable occurs only once by undoing operations. If the variable occurs more than once, however, other methods are needed. The balancing metaphor (doing the same thing to both sides) is useful for solving a variety of equations. In this lesson it's used for solving linear equations.

Guiding the Investigation

One Step

Show the Pan Balance transparency. Pose this problem: "How could you use pennies and cups on a pan balance to represent and solve the equation $2x$ plus 3 equals 7?" Be open to a variety of approaches. As needed, point out

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
	Geometry	✓	Communication
✓	Measurement	✓	Connections
	Data/Probability	✓	Representation

LESSON OBJECTIVE

- Learn the balancing method to solve equations by doing the same thing to both sides

Step 3 $x = 2$

Step 3 Redraw the picture, this one showing half of what was on each side of the scale in Step 2. There should be just one cup on the left side of the scale and the correct number of pennies on the right side needed to balance it. Write the equation that this picture represents. This is the solution to the original equation.

Now your group will create a pennies-and-cups equation for another group to solve.

Step 4 Divide the pennies into two equal piles. If you have one left over, put it aside. Draw a large equal sign (or form one with two pencils) and place the penny stacks on opposite sides of it.

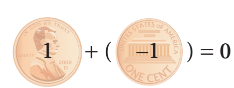
Step 5 From the pile on one side of your equal sign, make three identical stacks, leaving at least a few pennies out of the stacks. Hide each stack under a paper cup. You should now have three cups and some pennies on one side of your equal sign.

Step 6 On the other side you should have a pile of pennies. On both sides of the equal sign you have the same number of pennies, but on one side some of the pennies are hidden under cups. You can think of the two sides of the equal sign as being the two sides of a balance scale. Write an equation for this setup, using x to represent the number of pennies hidden under one cup.

Step 7 Move to another group's setup. Look at their arrangement of pennies and cups, and write an equation for it. Solve the equation; that is, find how many pennies are under one cup without looking. When you're sure you know how many pennies are under each cup, you can look to check your answer.

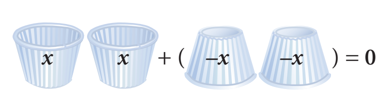
Step 8 Write a brief description of how you solved the equation.

You can do problems like those in the investigation using a balance scale as long as the weight of the cup is very small. But an actual balance scale can only model equations in which all the numbers involved are positive. Still, the idea of balancing equations can apply to equations involving negative numbers. Just remember, when you add any number to its opposite, you get 0. For this reason, the opposite of a number is called the **additive inverse.** Think of negative and positive numbers as having opposite effects on a balance scale. You can remove 0 from either side of a balance-scale picture without affecting the balance. These three figures all represent 0:



$1 + (-1) = 0$

$-3 + 3 = 0$



$2x + (-2x) = 0$

that an unbalanced scale doesn't tell as much as a balanced one. Remind students that to keep a balance they must do the same thing to both sides. If there's time, have each group set up an equation for other groups to solve.

Step 1 If students ask, point out that the balance pictured is a pan balance, with all masses on each side concentrated on one point, as opposed to the beam balance of Chapter 2, in which the distribution of masses was important.

Steps 7 and 8 Students may use undoing to solve the equation. Encourage them to see how working backward is used on one side of the balance to isolate the unknown.

SHARING IDEAS

At one station for groups to visit during Steps 4 through 8, you might place five pennies under each of two cups, lay out four pennies and seven washers on the same side, and put ten pennies and three washers on the other side. Explain to visiting students that each washer represents -1. Then, when the class is together, ask how they solved the equation $2x + 4 - 7 = 10 - 3$, or $2x - 3 = 7$. Elicit the idea that, just as they combined the four pennies with four washers and removed them, they could add three pennies to each side to "cancel out" the three washers remaining on the left. In other words, when you add a number to its opposite, you get 0, and you can remove 0's from anywhere on the balance without effect. (This wouldn't work on an actual pan balance, because the washers would have positive weights.)

If you don't have washers, ask during Sharing how you might model the equation $2x - 3 = 7$ on a pan balance. Students may suggest that you could add three pennies to the seven on the right. Point out that doing so models the equation $2x = 7 + 3$. **[Ask]** "Do the equations $2x - 3 = 7$ and $2x = 7 + 3$ have the same solution?" [yes]

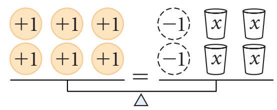
For more practice with balancing, use the Sketchpad demonstration The Balancing Method.

EXAMPLE A

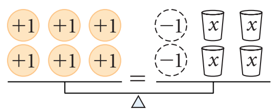
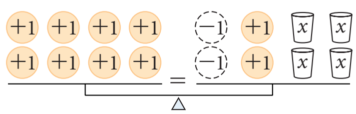
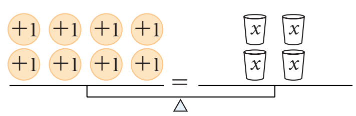
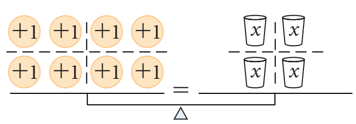
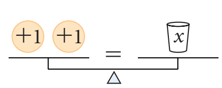
Draw a series of balance-scale pictures to solve the equation $6 = -2 + 4x$.

▶ Solution

The goal is to end up with a single x-cup on one side of the balance scale. One way to get rid of something on one side is to add its opposite to both sides.



Here is the equation $6 = -2 + 4x$ solved by the balancing method:

Picture	Action taken	Equation
	Original equation.	$6 = -2 + 4x$
	Add 2 to both sides.	$6 + 2 = -2 + 2 + 4x$
	Remove the 0.	$8 = 4x$
	Divide both sides by 4.	$\frac{8}{4} = \frac{4x}{4}$
	Reduce.	$2 = x$ or $x = 2$

In the second and third equations, you saw $6 + 2$ combine to 8, and $-2 + 2$ combine to 0. You can combine numbers because they are *like terms*. However, in the first equation you could not combine -2 and $4x$, because they are *not* like terms. **Like terms** are terms in which the variable component is the same, and they may differ only by a coefficient.

Assessing Progress

Observe how well students follow directions, work in groups, and understand the idea of balance.

▶ EXAMPLE A

This example continues the idea of working with negative numbers on a picture of a pan balance (not an actual balance). You might want to reinforce the new vocabulary *additive inverse* as you discuss this example. Exercise 6 defines *multiplicative inverse*.

If you decide to show students an example of an equation in which x has a negative coefficient, be aware of the pitfalls of the balancing model. You may use undoing and divide or multiply by the multiplicative inverse.

Another option is to move the term to the other side of the equation to give it a positive coefficient.

▶ EXAMPLE B

This example illustrates how to solve a linear equation by four different methods. With an appropriate choice of window, the calculator methods can be made more exact, but in general finding that window is more difficult than using the other methods.

Balance-scale pictures can help you see what to do to solve an equation by the balancing method. But you won't need the pictures once you get the idea of doing the same thing to both sides of an equation. And pictures are less useful if the numbers in the equation aren't "nice."

EXAMPLE B

Solve the equation $-31 = -50.25 + 1.55x$ using each method.

a. undoing operations

b. the balancing method

c. tracing on a calculator graph

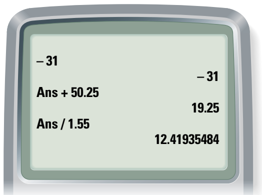
d. zooming in on a calculator table

▶ Solution

Each of these methods will give the same answer, but notice the differences among the methods. When might you prefer to use a particular method?

a. undoing operations

Start with -31.



b. the balancing method

$-31 = -50.25 + 1.55x$	Original equation.
$-31 + 50.25 = -50.25 + 50.25 + 1.55x$	Add 50.25 to both sides.
$19.25 = 1.55x$	Combine like terms. (Evaluate and remove the 0.)
$\frac{19.25}{1.55} = \frac{1.55x}{1.55}$	Divide both sides by 1.55.
$12.42 \approx x$, or $x \approx 12.42$	Reduce.

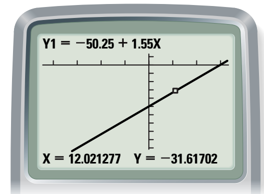
This chart shows how balancing equations is related to the undoing method that you've been using. In the last column, as you work up from the bottom, you can see how the equation changes as you apply the "undo" operation to both sides of the equation.

Equation: $-31 = -50.25 + 1.55x$			
Description	**Undo**	**Result**	**Equation**
Pick x.		≈ 12.42	$12.42 \approx x$
Multiply by 1.55.	/ (1.55)	19.25	$19.25 = 1.55x$
Subtract 50.25.	+ (50.25)	-31	$-31 = -50.25 + 1.55x$

In parts a and b, if you convert the answer to a fraction, you get an exact solution of $\frac{385}{31}$.

c. tracing on a calculator graph

Enter the equation into Y1. Adjust your window settings and graph. Press TRACE and use the arrow keys to find the x-value for a y-value of -31. (See Example B in Lesson 3.4 to review this procedure.) You can see that for a y-value of approximately -31.6 the x-value is 12.02.

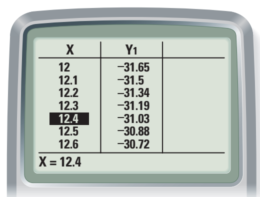


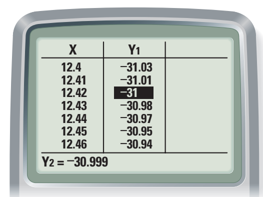
[−55, 35, 10, −120, 30, 10]

d. zooming in on a calculator table

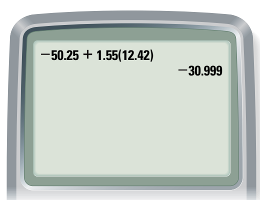
To find a starting value for the table, use guess-and-check or a calculator graph to find an approximate answer. Then use the calculator table to find the answer to the desired accuracy.

Once you have determined a reasonable starting value, zoom in on a calculator table to find the answer using smaller and smaller values for the table increment. [▶ See **Calculator Note 2A** to review zooming in on a table. ◀]





You can also check your answer by using substitution.



The calculator result isn't exactly -31 because 12.42 is a rounded answer. If you substitute an exact solution such as $\frac{19.25}{1.55}$ or $\frac{385}{31}$, you'll get exactly -31.

From Example B, you can see that each method has its advantages. The methods of balancing and undoing use the same process of working backward to get an exact solution. The two calculator methods are easy to use but usually give approximate solutions to the equation. You may prefer one method over others, depending on the equation you need to solve. If you are able to solve an equation using two or more different methods, you can check to see that each method gives the same result. With practice, you may develop symbolic solving methods of your own. Knowing a variety of methods, such as the balancing and undoing methods, as well as the calculator methods, will improve your equation-solving skills, regardless of which method you prefer.

In Exercise 12, you'll see how to use the balancing method to solve an equation that has the variable on both sides.

Closing the Lesson

Of the two exact methods for solving linear equations, the **balancing method** is useful when the variable appears more than once in the equation. The undoing method helps students focus on the order of operations.

BUILDING UNDERSTANDING

Students practice solving equations by a variety of methods.

ASSIGNING HOMEWORK

Essential	1–6, 9, 12
Performance assessment	9
Portfolio	9
Journal	6
Group	8, 10–12
Review	13–15

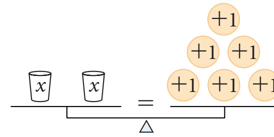
EXERCISES

You will need your graphing calculator for Exercises **8** and **14.**

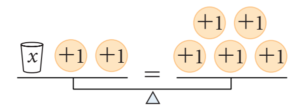
Practice Your Skills

1. Give the equation that each picture models.

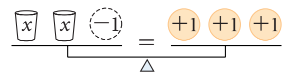
a. @ $2x = 6$



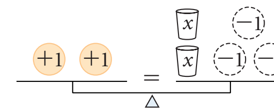
b. $x + 2 = 5$



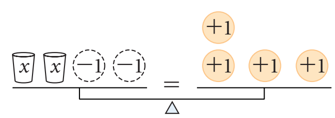
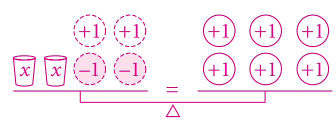
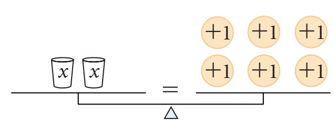
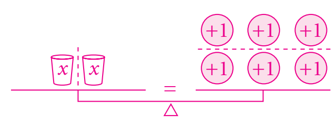
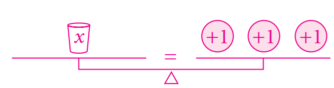
c. $2x - 1 = 3$



d. $2 = 2x - 3$



2. Copy and fill in the table to solve the equation as in Example A.

Picture	Action taken	Equation
	Original equation.	$2x - 2 = 4$
	Add 2 to both sides.	$2x - 2 + 2 = 4 + 2$
	Remove 0 from left side.	$2x = 6$
	Divide both sides by 2.	$\frac{2x}{2} = \frac{6}{2}$
	Reduce.	$x = 3$

3. Give the next stages of the equation, matching the action taken, to reach the solution.

a. $0.1x + 12 = 2.2$ @

Equation	Action
$0.1x + 12 = 2.2$	Original equation.
$0.1x + 12 - 12 = 2.2 - 12$	Subtract 12 from both sides.
$0.1x = -9.8$	Remove the 0 and subtract.
$x = -98$	Divide both sides by 0.1.

b. $\frac{12 + 3.12x}{3} = -100$

Equation	Action
$\frac{12 + 3.12x}{3} = -100$	Original equation.
$12 + 3.12x = -300$	Multiply both sides by 3.
$12 - 12 + 3.12x = -300 - 12$	Subtract 12 from both sides.
$3.12x = -312$	Remove the 0.
$x = -100$	Divide both sides by 3.12

4. Complete the tables to solve the equations.

a.

Equation: $\frac{3(x-8)}{5} + 7 = 34$			
Description	**Undo**	**Result**	**Equation**
Pick x.		53	$x = 53$
Subtract 8.	$+ (8)$	45	$x - 8 = 45$
Multiply by 3.	$/ (3)$	135	$3(x - 8) = 135$
Divide by 5.	$\cdot (5)$	27	$\frac{3(x-8)}{5} = 27$
Add 7.	$- (7)$	34	$\frac{3(x-8)}{5} + 7 = 34$

b.

Equation: $7\left(\frac{2+x}{4}\right) - 5 = 16$			
Description	**Undo**	**Result**	**Equation**
Pick x.		10	$x = 10$
Add 2.	$- (2)$	12	$2 + x = 12$
Divide by 4.	$\cdot (4)$	3	$\frac{2+x}{4} = 3$
Multiply by 7.	$/ (7)$	21	$7\left(\frac{2+x}{4}\right) = 21$
Subtract 5.	$+ (5)$	16	$7\left(\frac{2+x}{4}\right) - 5 = 16$

5. Give the additive inverse of each number.

a. $\frac{1}{5}$ @ $-\frac{1}{5}$ **b.** 17 -17 **c.** -2.3 2.3 **d.** $-x$ x

Reason and Apply

6. A **multiplicative inverse** is a number or expression that you can multiply by something to get a value of 1. The multiplicative inverse of 4 is $\frac{1}{4}$ because $4 \cdot \frac{1}{4} = 1$. Give the multiplicative inverse of each number.

a. 12 @ $\frac{1}{12}$ **b.** $\frac{1}{6}$ 6 **c.** 0.02 50 **d.** $-\frac{1}{2}$ -2

7. Solve these equations. Tell what action you take at each stage.

a. $144x = 12$ $x = \frac{1}{12}$ **b.** $\frac{1}{6}x + 2 = 8$ $x = 36$

8. *Mini-Investigation* A solution to the equation $-10 + 3x = 5$ is shown below.

$$-10 + 3x = 5$$
$$3x = 15$$
$$x = 5$$

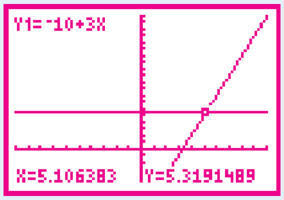
a. Describe the steps that transform the original equation into the second equation and the second equation into the third (the solution). Add 10 to both sides, divide both sides by 3.

b. Graph $Y_1 = -10 + 3x$ and $Y_2 = 5$, and trace to the lines' intersection. Write the coordinates of this point. (5, 5)

Helping with the Exercises

Exercise 7 Most students will divide both sides of 7a by 144. For 7b, students may multiply both sides by 6, then subtract 12 from both sides, or they may subtract 2 from both sides, then multiply both sides by 6.

8b.



[−10, 10, 1, −5, 20, 1]

c. Graph Y1 $= 3x$ and Y2 $= 15$, and trace to the lines' intersection. Write the coordinates of this point. (5, 15)

d. Graph Y1 $= x$ and Y2 $= 5$, and trace to the lines' intersection. Write the coordinates of this point. (5, 5)

e. What do you notice about your answers to 8b–d? Explain what this illustrates.

9. Solve the equation $4 + 1.2x = 12.4$ by using each method.

a. balancing **b.** undoing **c.** tracing on a graph **d.** zooming in on a table

10. Solve each equation symbolically using the balancing method.

a. $3 + 2x = 17$ @ **b.** $0.5x + 2.2 = 101.0$ **c.** $x + 307.2 = 2.1$

d. $2(2x+2) = 7$ **e.** $\dfrac{4 + 0.01x}{6.2} - 6.2 = 0$ @

11. You can solve familiar formulas for a specific variable. For example, solving $A = lw$ for l you get

$A = lw$	Original equation.
$\dfrac{A}{w} = \dfrac{lw}{w}$	Divide both sides by w.
$\dfrac{A}{w} = l$	Reduce.

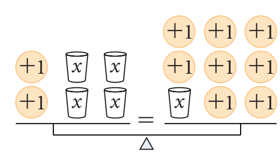
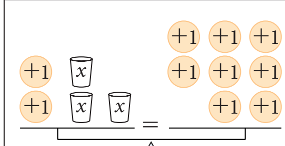
You can also write $l = \frac{A}{w}$. Now try solving these formulas for the given variable.

a. $C = 2\pi r$ for r @ $r = \dfrac{C}{2\pi}$ **b.** $A = \frac{1}{2}(hb)$ for h $h = \dfrac{2A}{b}$ **c.** $P = 2(l + w)$ for l @ $l = \dfrac{P}{2} - w$

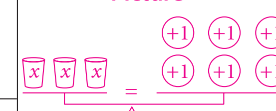
d. $P = 4s$ for s $s = \dfrac{P}{4}$ **e.** $d = rt$ for t $t = \dfrac{d}{r}$ **f.** $A = \frac{1}{2}h(a + b)$ for h $h = \dfrac{2A}{a + b}$

12. An equation can have the variable on both sides. In these cases you can maintain the balance by eliminating the x's from one of the sides before you begin undoing.

a. Copy and complete this table to solve the equation. @

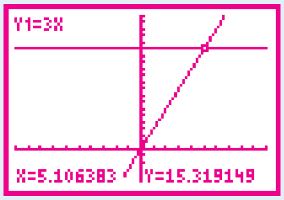
Picture	Action taken	Equation
	Original equation.	$2 + 4x = x + 8$
	Subtract $1x$ from both sides.	$2 + 3x = 8$
See picture at right.	Subtract 2 from both sides.	$3x = 6$
See picture at right.	Divide both sides by 3.	$x = 2$

Picture

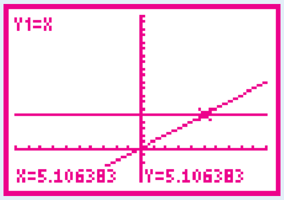


b. Show the steps used to solve $5x - 4 = 2x + 5$ using the balancing method. Substitute your solution into the original equation to check your answer.

8c.



8d.



8e. Even though the lines are different in each graph, in all three graphs the x-coordinate of the intersection is the same: $x = 5$. This illustrates that transforming the equation by doing the same thing to both sides does not change the solution.

Exercise 9 There are different ways to keep the equation balanced as it is solved. For example, the first step could be to multiply both sides by 10.

Exercise 11 As needed, remind students that lw means the product of l and w and that the answer gives the length of a rectangle in terms of its area and width. Although familiarity with the other formulas is not necessary in order to solve the equations, you might ask what the other formulas are used for. [a. circumference of a circle; b. area of a triangle; c. perimeter of a rectangle; d. perimeter of a square; e. distance traveled at a constant rate; f. area of a trapezoid] In 11f, students might have trouble isolating h. As a hint, you can suggest treating $(a + b)$ as a single number.

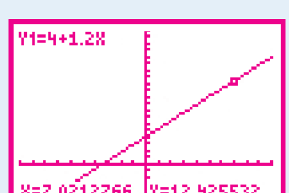
9a.

$4 + 1.2x = 12.4$	Original equation.
$4 - 4 + 1.2x = 12.4 - 4$	Subtract 4 from both sides.
$1.2x = 8.4$	Remove the 0 and subtract.
$\dfrac{1.2x}{1.2} = \dfrac{8.4}{1.2}$	Divide both sides by 1.2.
$x = 7$	Reduce.

9b.

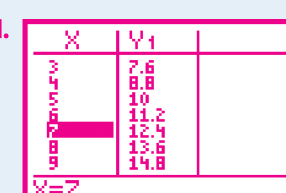
Start with 12.4.	12.4
Ans − 4	8.4
Ans/1.2	7

9c.



[−10, 10, 1, −5, 20, 1]

9d.



Review

2.1 **13.** **APPLICATION** Economy drapes for a certain size window cost \$90. They have shallow pleats, and the width of the fabric is $2\frac{1}{4}$ times the window width. Luxury drapes of the same fabric for the same size window have deeper pleats. The width of the fabric is 3 times the window width. What price should the store manager ask for the luxury drapes? ⓗ $\frac{\$90}{2.25} = \frac{x}{3}$, $x = \$120$

3.4 **14.** Run the easy level of the LINES program on your calculator. [▶ See **Calculator Note 3D** to learn how to use the LINES program. ◀] Sketch a graph of the randomly generated line on your paper. Use the trace function to locate the y-intercept and to determine the rate of change. When the calculator says you have the correct equation, write it under the graph. Repeat this program until you get three correct equations in a row.

3.2 **15.** The local bagel store sells a baker's dozen of bagels for \$6.49, while the grocery store down the street sells a bag of 6 bagels for \$2.50.

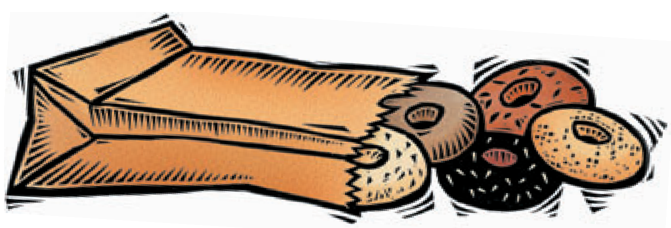
a. Copy and complete the tables showing the cost of bagels at the two stores.

Bagel Store

Bagels	13	26	39	52	65	78
Cost	6.49	12.98	19.47	25.96	32.45	38.94

Grocery Store

Bagels	6	12	18	24	30	36	42	48	54	60
Cost	2.50	5.00	7.50	10.00	12.50	15.00	17.50	20.00	22.50	25.00



b. Graph the information for each market on the same coordinate axes. Put bagels on the horizontal axis and cost on the vertical axis.

c. Find equations to describe the cost of bagels at each store.

d. How much does one bagel cost at each store? How do these cost values relate to the equations you wrote in 15c?

e. Looking at the graphs, how can you tell which store is the cheaper place to buy bagels? the grocery store, because its line is lower

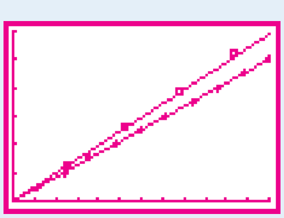
f. Bernie and Buffy decided to use a recursive routine to complete the tables. Bernie used this routine for the bagel store:

6.49 ENTER

Ans · 2 ENTER

Buffy says that this routine isn't correct, even though it gives the correct answer for 13 and 26 bagels. Explain to Bernie what is wrong with his recursive routine. What routine should he use? Bernie's routine calculates each price by doubling the last. It works the first time, because if you buy twice as many bagels, you pay twice as much. But using Bernie's routine, if you buy 36 bagels at the bagel store, you pay \$25.96 instead of \$19.47, which amounts to paying four times as much as a single dozen instead of three times the price of a dozen. The routine should be: 6.49 ENTER, Ans + 6.49, ENTER, ENTER, ...

15b.



[0, 72, 6, 0, 30, 5]

The line with the square markers is the bagel store, and the line with the crosses is the grocery store.

15c. y represents cost; x represents number of bagels.

bagel store: $y = \frac{6.49}{13}x$ (or $y \approx 0.50x$)

grocery store: $y = \frac{2.50}{6}x$ (or $y \approx 0.42x$)

15d. Bagel store: about 50¢ per bagel; grocery store: about 42¢ per bagel; these are the coefficients of x or constants of variation in the equations.

10a. $3 + 2x = 17$

$$3 - 3 + 2x = 17 - 3$$
$$2x = 14$$
$$\frac{2x}{2} = \frac{14}{2}$$
$$x = 7$$

10b. $0.5x + 2.2 = 101.0$

$$0.5x + 2.2 - 2.2 = 101.0 - 2.2$$
$$0.5x = 98.8$$
$$\frac{0.5x}{0.5} = \frac{98.8}{0.5}$$
$$x = 197.6$$

10c. $x + 307.2 = 2.1$

$$x + 307.2 - 307.2 = 2.1 - 307.2$$
$$x = -305.1$$

10d. $2(2x + 2) = 7$

$$\frac{2(2x + 2)}{2} = \frac{7}{2}$$
$$2x + 2 = 3.5$$
$$2x + 2 - 2 = 3.5 - 2$$
$$2x = 1.5$$
$$\frac{2x}{2} = \frac{1.5}{2}$$
$$x = 0.75$$

10e. $\frac{4 + 0.01x}{6.2} - 6.2 = 0$

$$\frac{4 + 0.01x}{6.2} - 6.2 + 6.2 = 0 + 6.2$$
$$\frac{4 + 0.01x}{6.2} = 6.2$$
$$\frac{4 + 0.01x}{6.2} \cdot 6.2 = 6.2 \cdot 6.2$$
$$4 + 0.01x = 38.44$$
$$4 - 4 + 0.01x = 38.44 - 4$$
$$0.01x = 34.44$$
$$\frac{0.01x}{0.01} = \frac{34.44}{0.01}$$
$$x = 3444$$

12b. $5x - 4 = 2x + 5$

$$5x - 2x - 4 = 2x - 2x + 5$$
$$3x - 4 = 5$$
$$3x - 4 + 4 = 5 + 4$$
$$3x = 9$$
$$x = 3$$

Check: $5(3) - 4 \stackrel{?}{=} 2(3) + 5$

$$15 - 4 \stackrel{?}{=} 6 + 5$$
$$11 = 11$$

Exercise 15 As needed, remind students that a *baker's dozen* is 13.

LESSON
3.7

PLANNING

LESSON OUTLINE

One day:

30 min Activity

20 min Improving Reasoning Skills

MATERIALS

- pieces of rope of different lengths (around 1 m) and thickness (two per group)
- metersticks or tape measures (two per group)
- Fathom demonstration Tying Knots, *optional*

TEACHING

Students experiment with the slope and y-intercept of a line that fits real-world data.

Guiding the Activity

This activity works best with one pair of students working on each rope. The Fathom demonstration Tying Knots can be used for Steps 3–9 of the activity.

One Step
Pose this problem: "Tie up to six knots in each rope, and predict the length of 10 m of rope after making 17 knots in each."

Step 1 **[ELL]** The *length of the rope* after knots are tied means the distance from one end to the other when the rope is stretched tight. If students want to use a knot other than an overhand knot, encourage the creativity, but ask them to make all the knots the same and to be sure there's enough rope to tie at least six of those knots.

Step 3 Be sure students realize that the rate of change is negative.

LESSON
3.7

Activity Day

Modeling Data

Whenever measuring is involved in collecting data, you can expect some variation in the pattern of data points. Usually, you can't construct a mathematical model that fits the data exactly. But in general, the better a model fits, the more useful it is for making predictions or drawing conclusions from the data.

Activity
Tying Knots

You will need
- two pieces of rope of different lengths (around 1 m) and thickness
- a meterstick or a tape measure

In this activity you'll explore the relationship between the number of knots in a rope and the length of the rope and write an equation to model the data.

Number of knots	Length of knotted rope (cm)
0	
1	
2	

Step 1 The length of rope should be reduced the same amount for each knot tied.

Step 2 The data should show a linear relationship.

Step 3 Answers will vary depending on the thickness of rope and type of knot. The rate of change represents the reduction of rope length for each knot tied. It is a negative number.

Step 1 Choose one piece of rope and record its length in a table like the one shown. Tie 6 or 7 knots, remeasuring the rope after you tie each knot. As you measure, add data to complete a table like the one above.

Step 2 Graph your data, plotting the number of knots on the x-axis and the length of the knotted rope on the y-axis. What pattern does the data seem to form?

Step 3 What is the approximate rate of change for this data set? What is the real-world meaning of the rate of change? What factors have an effect on it?

Step 4 What is the y-intercept for the line that best models the data? What is its real-world meaning? The y-intercept is the length of rope without any knots.

LESSON OBJECTIVES

- Find an equation that fits a set of real-world data
- Use a mathematical model to make predictions

NCTM STANDARDS

CONTENT		PROCESS	
✓	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
	Geometry		Communication
✓	Measurement	✓	Connections
	Data/Probability	✓	Representation

Step 5 Write an equation in intercept form for the line that you think best models the data. Graph your equation to check that it's a good fit. Graphs will vary. The line should go down from a positive y-intercept. It should pass through, or near, most of the points.

Now you'll make predictions and draw some conclusions from your data using the line model as a summary of the data.

Step 6 Use your equation to predict the length of your rope with 7 knots. What is the difference between the actual measurement of your rope with 7 knots and the length you predicted using your equation? Answers will vary.

Step 7 Use your equation to predict the length of a rope with 17 knots. Explain the problems you might have in making or believing your prediction.

Step 7 The equation says that you will eventually have a rope of length 0 if you tie enough knots.

Step 8 What is the maximum number of knots that you can tie with your piece of rope? Explain your answer.

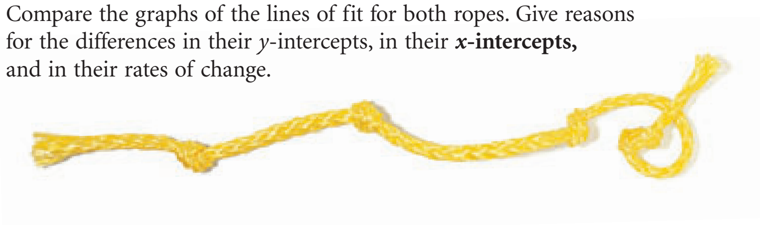
Step 9 Does your graph cross the x-axis? Explain the real-world meaning, if any, of the x-value of the intersection point.

Step 10 Substitute a value for y into the equation. What question does the equation ask? What is the answer? The question it asks is "How many knots are tied to produce a rope of length y?"

Step 11 Repeat Steps 1–5 using a different piece of rope. Graph the data on the same pair of axes.

Step 12 Compare the graphs of the lines of fit for both ropes. Give reasons for the differences in their y-intercepts, in their x-**intercepts,** and in their rates of change.

Step 12 Different lengths of ropes will account for different y-intercepts. Different thicknesses will account for different rates of change. Both affect the value of the x-intercepts.



IMPROVING YOUR REASONING SKILLS

There are 100 students and 100 lockers in a school hallway. All of the lockers are closed. The first student walks down the hallway and opens every locker. A second student closes every even-numbered locker. The third student goes to every third locker and opens it if it is closed or closes it if it is open. This pattern repeats so that the nth student leaves every nth locker the opposite of how it was before. After all 100 students have opened or closed the lockers, how many lockers are left open?

Step 4 If students want to graph various lines to find the rate of change and y-intercept of the line that seems to fit the data best, encourage them. This experimentation with slope and intercept can give them a good feel for both.

Step 5 As needed, explain that a line is a good fit to a data set if it goes pretty close to each data point and there are about as many points above the line as below the line. Students will see a more systematic way of finding lines of good fit in Lessons 4.6 and 4.7.

Step 8 Answers will vary. Students will get to a point where they're tying knots on top of knots.

Step 9 The equation will cross the x-axis, but the points will not. It is not possible to have zero rope length no matter how many knots are tied. Even the knot itself accounts for some length.

Step 12 **[Ask]** "Does the thickness of the rope itself have any bearing on the results? Does the type of knot affect the results?"

SHARING IDEAS

Pick out any unusual approaches for sharing.

Assessing Progress
Observe students' understanding of how y-intercept and rate of change affect the graph of a line.

IMPROVING REASONING SKILLS

After accumulating data, many students will see the pattern that the lockers left open at the end correspond to perfect squares—1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. If needed, explain that *left open* means open after all 100 students pass. Be sure they actually answer the question of how many lockers are left open. [10] Ask students to explain. [A locker changes once for each factor of its number. For example, locker number 24 is changed by students 1, 2, 3, 4, 6, 8, 12, and 24. So if a locker's number has an even number of factors, it is left closed. If a locker's number has an odd number of factors, it is left open. The numbers with an odd number of factors are the perfect squares.] **[Ask]** "Why do perfect squares have an odd number of factors?" [Factors of numbers come in pairs: (1, 24), (2, 12), (3, 8), (4, 6). The square root of a perfect square is paired with itself; factors of 36: (1, 36), (2, 18), (3, 12), (4, 9), (6, 6). So the number of distinct factors is an odd number.]

PLANNING

LESSON OUTLINE

One day:

5 min	Introduction
15 min	Exercises and helping individuals
15 min	Checking work and helping individuals
15 min	Student self-assessment

REVIEWING

Direct students' attention to the table of floor heights in Lesson 3.1, Example A. **[Ask]** "What linear equation describes the floor heights?" [$y = -4 + 13x$; this is intercept form.] Then **[Ask]** "What floor has height 282 ft?" To answer this, review how to set up the linear equation $-4 + 13x = 282$ and ask how to solve it. Bring out the ideas of generating the table recursively, undoing, and balancing. If needed, review rules for order of operations.

As students work individually on these exercises, you can work more with individual students who have been having difficulties. The Mixed Review contains problems from Chapters 1–3.

ASSIGNING HOMEWORK

Assigning either the evens or the odds will give students a good review. They could work on the other exercises in groups.

Helping with the Exercises

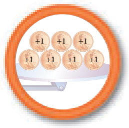
Exercise 1 Reasons students give for each step will depend on their method of solving the equation.

CHAPTER 3 REVIEW

You started this chapter by investigating **recursive sequences** by using their starting values and **rates of change** to write **recursive routines.** You saw how rates of change and starting values appear in plots.

In a walking investigation you observed, interpreted, and analyzed graphical representations of relationships between time and distance. What does the graph look like when you stand still? When you move away from or move toward the motion sensor? If you speed up or slow down? You identified real-world meanings of the ***y*-intercept** and the rate of change of a **linear relationship,** and used them to write a **linear equation** in the **intercept form,** $y = a + bx$. You learned the role of b, the coefficient of x. You explored relationships among verbal descriptions, tables, recursive rules, equations, and graphs.

Throughout the chapter you developed your equation-solving skills. You found solutions to equations by continuing to practice an undoing process and by using a **balancing** process. You found approximate solutions by tracing calculator graphs and by zooming in on calculator tables. Finally, you learned how to model data that don't lie exactly on a line, and you used your model to predict inputs and outputs.



EXERCISES

You will need your graphing calculator for Exercises **4, 6,** and **7.**

@ Answers are provided for all exercises in this set.

1. Solve these equations. Give reasons for each step.

a. $-x = 7$ $x = -7$

b. $4.2 = -2x - 42.6$ $x = -23.4$

2. These tables represent linear relationships. For each relationship, give the rate of change, the y-intercept, the recursive rule, and the equation in intercept form.

a.

x	y
0	3
1	4
2	5

1; 3; add 1; $y = 3 + x$

b.

x	y
1	0.01
2	0.02
3	0.03

0.01; 0; add 0.01; $y = 0.01x$

c.

x	y
−2	1
0	5
3	11

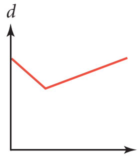
2; 5; add 2; $y = 5 + 2x$

d.

x	y
−4	5
12	−3
2	2

$-\frac{1}{2}$; 3; subtract $\frac{1}{2}$; $y = 3 - \frac{1}{2}x$

3. Match these walking instructions with their graph sketches.

i.


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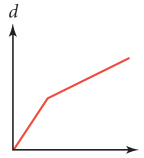
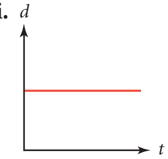


iii.


a. The walker stands still. iii

b. The walker takes a few steps toward the 0-mark, then walks away. i

c. The walker steps away from the 0-mark, stops, then continues more slowly in the same direction. ii

4. Graph each equation on your calculator, and trace to find the approximate y-value for the given x-value.

a. $y = 1.21 - x$ when $x = 70.2$ $y = -68.99$

b. $y = 6.02 + 44.3x$ when $x = 96.7$ $y = 4289.83$

c. $y = -0.06 + 0.313x$ when $x = 0.64$ $y = 0.14032$

d. $y = 1183 - 2140x$ when $x = -111$ $y = 238{,}723$

5. Write the equations for linear relationships that have these characteristics.

a. The output value is equal to the input value. $y = x$

b. The output value is 3 less than the input value. $y = -3 + x$

c. The rate of change is 2.3 and the y-intercept is -4.3. $y = -4.3 + 2.3x$

d. The graph contains the points (1, 1), (2, 1), and (3, 1). $y = 1$

6. The profit for a small company depends on the number of bookcases it sells. One way to determine the profit is to use a recursive routine such as

{0, −850} ENTER

{Ans(1) + 1, Ans(2) + 70} ENTER; ENTER, . . .

a. Explain what the numbers and expressions 0, −850, Ans(1), Ans(1) + 1, Ans(2), and Ans(2) + 70 represent.

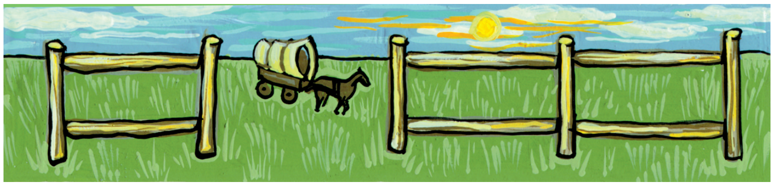
b. Make a plot of this situation.

c. When will the company begin to make a profit? Explain.

d. Explain the relationship between the values −850 and 70 and your graph.

e. Does it make sense to connect the points in the graph with a line? Explain.
No; partial bookcases cannot be sold.

7. A single section and a double section of a log fence are shown.



a. How many additional logs are required each time the fence is increased by a single section? 3

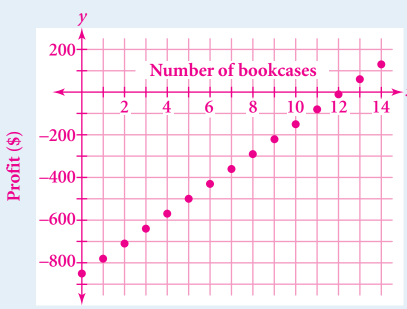
6a. 0 represents no bookcases sold; −850 represents fixed overhead, such as start-up costs; Ans(1) represents the previously calculated number of bookcases sold; Ans(1) + 1 represents the current number of bookcases sold, one more than the previous; Ans(2) represents the profit for the previous number of bookcases; Ans(2) + 70 represents the profit for the current number of bookcases—the company makes $70 more profit for each additional bookcase sold.

Exercise 6c Have the students create a table from their data. **[Ask]** "What is the value of x when $f(x) = 0$?" [$x \approx 12.1$] Relate this value (the zero of the function) to the answer found.

6c. Sample answer: The graph crosses the x-axis at approximately 12.1 and is positive after that; the company needs to make at least 13 bookcases to make a profit.

6d. −850, the profit if the company makes zero bookcases, is the y-intercept; 70, the amount of additional profit for each additional bookcase, is the rate of change; y goes up by $70 each time x goes up by one bookcase.

6b.



b. Copy and fill in the missing values in the table below.

Number of sections	1	2	3	4	. . .	30	. . .	50
Number of logs	4	7	10	13	. . .	91	. . .	151

c. Describe a recursive routine that relates the number of logs required to the number of sections. 4 ENTER , Ans + 3 ENTER , ENTER , . . .

d. If each section is 3 meters long, what is the longest fence you can build with 217 logs? 216 m

8. Suppose a new small-business computer system costs \$5,400. Every year its value drops by \$525.

a. Define variables and write an equation modeling the value of the computer in any given year.

b. What is the rate of change, and what does it mean in the context of the problem?

c. What is the y-intercept, and what does it mean in the context of the problem?

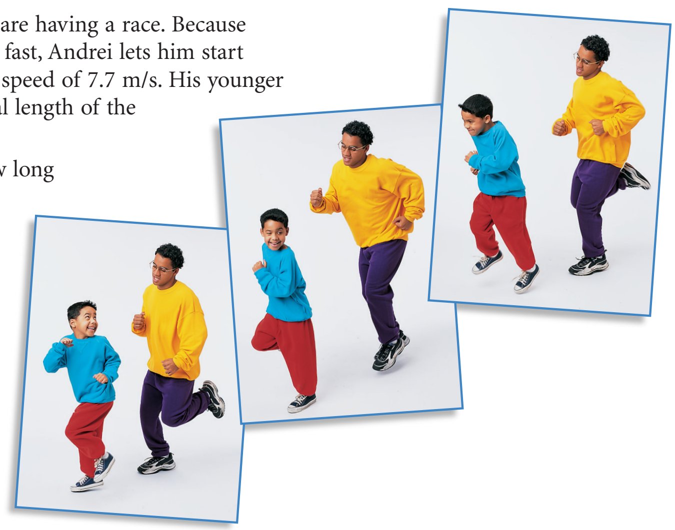
d. What is the x-intercept, and what does it mean in the context of the problem?

9. Andrei and his younger brother are having a race. Because the younger brother can't run as fast, Andrei lets him start out 5 m ahead. Andrei runs at a speed of 7.7 m/s. His younger brother runs at 6.5 m/s. The total length of the race is 50 m.

a. Write an equation to find how long it will take Andrei to finish the race. Solve the equation to find the time.

b. Write an equation to find how long it will take Andrei's younger brother to finish the race. Solve the equation to find the time.

c. Who wins the race? How far ahead was the winner at the time he crossed the finish line?



10. Solve each equation using the method of your choice. Then use a different method to verify your solution.

a. $14x = 63$ $x = 4.5$

b. $-4.5x = 18.6$ $x = -4.1\overline{3}$

c. $8 = 6 + 3x$ $x = 0.\overline{6}$

d. $5(x - 7) = 29$ $x = 12.8$

e. $3(x - 5) + 8 = 12$ $x = 6.\overline{3}$

8a. Let v represent the value in dollars and y represent the number of years; $v = 5400 - 525y$.

8b. The rate of change is -525; in each additional year, the value of the computer system decreases by \$525.

8c. The y-intercept is 5400; the original value of the computer system is \$5,400.

Exercise 8d Have the students enter their equations into the graphing calculator and examine the table. **[Ask]** "What is the value of x when $f(x) = 0$?" [$x \approx 10.3$] and "How does this value (the zero of the function) compare to the x-intercept of the graph?" [The value for which $f(x) = 0$ is the same as the point where the line crosses the x-axis.]

8d. The x-intercept is approximately 10.3; this means that the computer system no longer has value after approximately 10.3 yr.

9a. $50 = 7.7t$

$$t = \frac{50}{7.7} \approx 6.5 \text{ s}$$

9b. $50 = 5 + 6.5t$

$$t = \frac{50 - 5}{6.5} \approx 6.9 \text{ s}$$

9c. Andrei wins; when Andrei finishes, his younger brother is $50 - [5 + 6.5(6.5)] \approx 2.8$ m from the finish line.

Exercise 10 Students can also check their results by substituting them into the equation.

11. For each table, write a formula for list L_2 in terms of list L_1.

a.

L_1	L_2
0	−5.7
1	−3.4
2	−1.1
3	1.2
4	3.5
5	5.8

$L_2 = -5.7 + 2.3 \cdot L_1$

b.

L_1	L_2
−3	19
−1	3
0	−5
2	−21
5	−45
6	−53

$L_2 = -5 - 8 \cdot L_1$

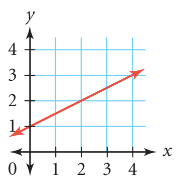
c.

L_1	L_2
3	13.5
−2	11
−9	7.5
0	12
6	15
−5	9.5

$L_2 = 12 + 0.5 \cdot L_1$

12. You can represent linear relationships with a graph, a table of values, an equation, or a rule stated in words. Here are two linear relationships. Give all the other ways to show each relationship.

a.



b.

x	y
−2	2
−1.5	1.5
0	0
3	−3

MIXED REVIEW

2.3 **13.** **APPLICATION** Sonja bought a pair of 210 cm cross-country skis. Will they fit in her ski bag, which is $6\frac{1}{2}$ ft long? Why or why not? No, they won't fit; 210 cm is 6.89 ft.

1.4 **14.** Fifteen students counted the number of letters in their first and last names. Here is the data set [Data set: **NMLET**]:

6	15	8	12	8	17	9	7
13	15	14	9	16	15	10	

a. Make a histogram of the data with a bin width of 2.

b. What is the mean number of letters? 11.6 letters

2.7 **15.** Evaluate these expressions.

a. $-3 \cdot 8 - 5 \cdot 6$ -54

b. $[-2 - (-4)] \cdot 8 - 11$ 5

c. $7 \cdot 8 + 4 \cdot (-12)$ 8

d. $11 - 3 \cdot 9 - 2$ -18

3.2 **16.** On a recent trip to Detroit, Tom started from home, which is 12 miles from Traverse City. After 4 hours he had traveled 220 miles.

a. Write a recursive routine to model Tom's distance from Traverse City during this trip. State at least two assumptions you're making.

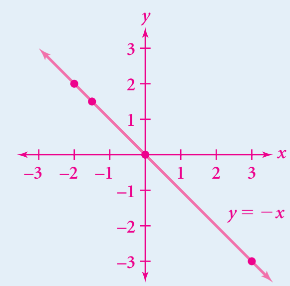
b. Use your recursive routine to determine his distance from Traverse City for each hour during the first 5 hours of the trip.

c. What is the rate of change, and what does it mean in the context of this situation?

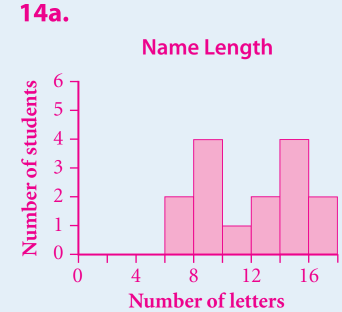
12a. $y = 1 + \frac{1}{2}x$; the output value is half the input value plus 1.

x	y
0	1
1	1.5
2	2
3	2.5
4	3

12b. $y = -x$; the output value is the additive inverse (or opposite) of the input value, or the sum of the input value and the output value is 0.



14a.



Exercise 16a If students don't know what to assume about how Tom's home is related to Traverse City and Detroit, tell them to assume something they can explain in 15b.

16a. The starting value is 12; Ans + 55. Possible assumptions: Tom's home is 12 mi closer to Detroit than to Traverse City. He travels at a constant speed. We are measuring highway distance.

16b.

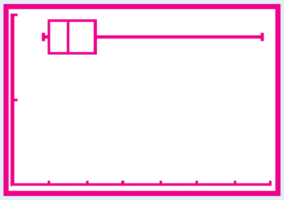
Hours	0	1	2	3	4	5
Distance (mi)	12	67	122	177	232	287

16c. Tom traveled 55 mi each additional hour. The rate of change is 55 mi/h.

17a. approximately 1061 thousand (or 1,0161,000) visitors

17b. 404, 482, 738, 1131, 3379

17c.



[0, 3500, 500, 0, 2, 1]

17d. Yosemite; the number of visitors exceeds 1131 by more than 1.5(1131 − 482).

1.3 **17.** California has many popular national parks. This table shows the number of visitors in thousands to national parks in 2003.

a. Find the mean number of visitors.

b. What is the five-number summary for the data?

c. Create a box plot for the data.

d. Identify any parks in California that are outliers in the numbers of visitors they had. Explain why they are outliers.

Park Attendance

National park	Visitors (thousands)
Channel Islands	586
Death Valley	890
Joshua Tree	1283
Kings Canyon	556
Lassen Volcanic	404
Redwood	408
Sequoia	979
Yosemite	3379

(*U.S. National Park Service*) [Data set: **CAPRK**]

Joshua Tree National Park, California

Lassen Volcanic National Park, California

2.5 **18.** Ohm's law states that electrical current is inversely proportional to the resistance. A current of 18 amperes is flowing through a conductor whose resistance is 4 ohms.

a. What is the current that flows through the system if the resistance is 8 ohms? 9 amperes

b. What is the resistance of the conductor if a current of 12 amperes is flowing? 6 ohms

Every knob or lever of this sound recording console regulates electric resistance in a current. The resistance varies directly with voltage and inversely with current.

2.8 **19.** Consider the equation $2(x - 6) = -5$.

a. Solve the equation. Solution methods will vary; $x = 3.5$.

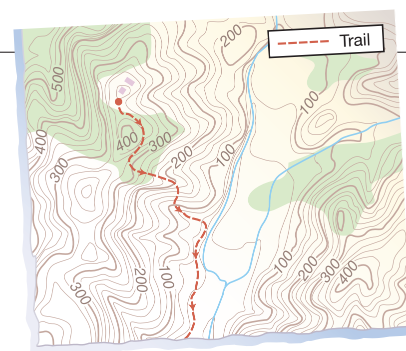
b. Show how you can check your result by substituting it into the original equation.
$2(3.5 - 6) = 2(-2.5) = -5$

2.3 **20.** **APPLICATION** Amber makes \$6 an hour at a sandwich shop. She wants to know how many hours she needs to work to save \$500 in her bank account. On her first paycheck, she notices that her net pay is about 75% of her gross pay.

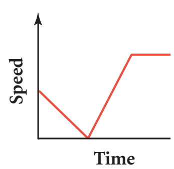
a. How many hours must she work to earn \$500 in gross pay? $\frac{500}{6} \approx 83.3$ h

b. How many hours must she work to earn \$500 in net pay? $\frac{500}{0.75 \cdot 6} \approx 111.1$ h

TAKE ANOTHER LOOK



1. The picture at right is a **contour map.** This type of map reveals the character of the terrain. All points on an **isometric line** are the same height in feet above sea level. The graph below shows how the hiker's walking speed changes as she covers the terrain on the dotted-line trail shown on the map.



a. What quantities are changing in the graph and in the map?

b. How does each display reveal rate of change?

c. How could you measure distance on each display?

d. What would the graph sketch of this hike look like if distance were plotted on the vertical axis instead of speed?

e. What do these two displays tell you when you study them together?

Sediment layers form contour lines in the Grand Canyon.

2. You've learned that a rational number is a number that can be written as a ratio of two integers. Every rational number can also be written in an equivalent decimal form. In Lesson 2.1, you learned how to convert fractions into decimal form. In some cases the result was a *terminating decimal,* and in other cases the result was a *repeating decimal,* in which a digit or group of digits repeated.

1e. The graph and the contour map together show that the speed of the hiker was decreasing at the beginning because she was climbing a hill. As she began to go down the hill, she kept moving faster and faster until she was running. She stayed at a constant run after she reached level ground.

Exercise 19 Students should solve by undoing. The distributive property will be introduced in Lesson 4.4.

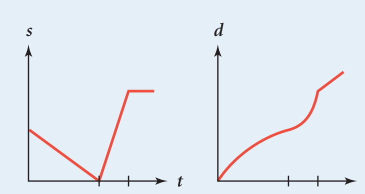
Take Another Look

1a. The graph shows the change in the hiker's speed over time. The contour map shows the change in the hiker's elevation as she follows the dotted line. It also shows the horizontal distance she has traveled from her starting point.

1b. The graph shows rate of change in speed as the steepness (slope) of a line. At first the speed is steadily decreasing, then it is increasing, then it remains constant. The contour map shows rate of change in elevation by the distance between the contour lines. When the lines are very close together, the elevation is changing quickly.

1c. If a scale were provided, you could infer distance from the graph by estimating the average speed up to a certain point and multiplying that by the time at that point. Students might refer to the formula *distance* equals *rate* (speed) times *time* ($d = rt$). You could measure distance on the contour map using the map scale.

1d. Student sketches of the graph of (*distance, time*) should show three sections. In each, the graph is increasing. In the first section, the graph is a curve that is concave down, showing a steadily decreasing speed. The second section of the graph is concave up because the speed is increasing, and the third section is a steep straight line, showing a constant, fast speed.



2a. 0.5, 0.4375, 0.088, $0.4\overline{6}$, $0.4\overline{09}$, $0.3\overline{6}$, 0.35

2b. When the fraction is in lowest terms, if the denominator only has factors of 2 and/or 5, the fraction will terminate.

2c. $\frac{25}{100}$ or $\frac{1}{4}$, $\frac{8}{10}$ or $\frac{4}{5}$, $\frac{13}{100}$, $\frac{412}{1000}$ or $\frac{103}{250}$

2d. $\frac{18}{99}$; multiply by 100.

2e. i. $\frac{32}{99}$ **ii.** $\frac{325}{999}$ **iii.** $\frac{2323}{9990}$

a. Rewrite each of these fractions in decimal form. If the digits appear to repeat, indicate this by placing a bar over those digits that repeat.

$$\frac{1}{2}, \frac{7}{16}, \frac{11}{125}, \frac{7}{15}, \frac{9}{22}, \frac{11}{30}, \frac{7}{20}$$

b. Describe how you can predict whether a fraction will convert to a terminating decimal or a repeating decimal.

Reversing the process—converting decimals to fractions

c. Write the decimals 0.25, 0.8, 0.13, and 0.412 as fractions.

You can use what you've learned in this chapter about solving equations to help you write an infinite repeating decimal, like $0.\overline{1}$, as a fraction. For example, to find a fraction equal to $0.\overline{1}$, you are looking for a fraction F such that $F = 0.11111\ldots$. Follow the steps shown.

$$F = 0.11111\ldots$$

$$10F = 1.11111\ldots$$

So, $10F - F = 1.11111\ldots - 0.11111\ldots$

$$9F = 1$$

$$F = \frac{1}{9}$$

Here, the trick was to multiply by 10 so that $10F$ and F had the same decimal part. Then, when you subtract $10F - F$, the decimal portion is eliminated.

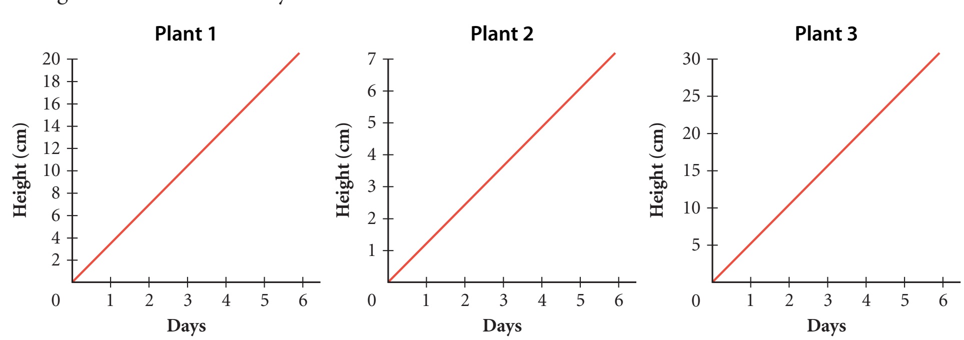
d. Write the repeating decimal $0.\overline{18}$ as a fraction. (*Hint:* What can you multiply $F = 0.\overline{18}$ by so that you can subtract off the same decimal part?)

e. Write these repeating decimals as fractions.

i. $0.\overline{32}$ **ii.** $0.\overline{325}$ **iii.** $0.2\overline{325}$

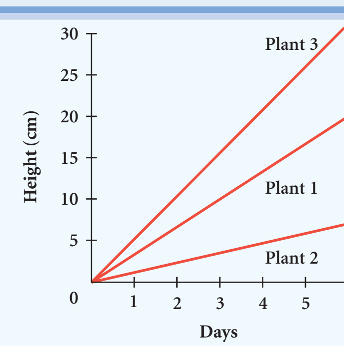
IMPROVING YOUR REASONING SKILLS

Did these plants grow at the same rate? If not, which plant was tallest on Day 4? Which plant took the most time to reach 8 cm? Redraw the graphs so that you can compare their growth rates more easily.



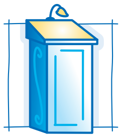
IMPROVING REASONING SKILLS

The differences in the vertical scale (height) indicate that the plants are growing at different rates. Plant 3 is the fastest growing; it is about 20 cm tall in 4 days. Plant 2 is slowest growing; it takes almost 6 days to reach 7 cm. This should suggest to students that the steepness of a line is relative to the scales on the axes. They have seen this many times on their calculators.



Assessing What You've Learned

GIVING A PRESENTATION



Making presentations is an important career skill. Most jobs require workers to share information, to help orient new coworkers, or to represent the employer to clients. Making a presentation to the class is a good way to develop your skill at organizing and delivering your ideas clearly and in an interesting way. Most teachers will tell you that they have learned more by trying to teach something than they did simply by studying it in school.

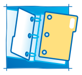
Here are some suggestions to make your presentation go well:

- Work with a group. Acting as a panel member might make you less nervous than giving a talk on your own. Be sure the role of each panel member is clear so that the work and the credit are equally shared.
- Choose the topic carefully. You can summarize the results of an investigation, do research for a project and present what you've learned and how it connects to the chapter, or give your own thinking on Take Another Look or Improving Your Reasoning Skills.
- Prepare thoroughly. Outline your presentation and think about what you have to say on each point. Decide how much detail to give, but don't try to memorize whole sentences. Illustrate your presentation with models, a poster, a handout, or overhead transparencies. Prepare these visual aids ahead of time and decide when to introduce them.
- Speak clearly. Practice talking loudly and clearly. Show your interest in the subject. Don't hide behind a poster or the projector. Look at the listeners when you talk.

Here are other ways to assess what you've learned:

UPDATE YOUR PORTFOLIO Choose a piece of work you did in this chapter to add to your portfolio—your graph from the investigation On the Road Again (Lesson 3.2), the most complicated equation you've solved, or your research on a project.

WRITE IN YOUR JOURNAL What method for solving equations do you like best? Do you always remember to define variables before you graph or write an equation? How are you doing in algebra generally? What things don't you understand?



ORGANIZE YOUR NOTEBOOK You might need to update your notebook with examples of balancing to solve an equation, or with notes about how to trace a line or search a table to approximate the coordinates of the solution. Be sure you understand the meanings of important words like linear equation, rate of change, and intercept form.

ASSESSING

You can use either Form A or Form B of the Chapter Test, or you can use Constructive Assessment items. Using the Test Generator CD, you can create an alternate version of the test or combine some items from Form A or Form B with Constructive Assessment items.

You may also use the Mixed Review as preparation for an exam that covers the material through Chapter 3. Two forms of the Chapters 1 to 3 Exam are available in the Assessment Resources.

FACILITATING SELF-ASSESSMENT

During Sharing, students have had some opportunity to present ideas to the class. In a presentation, they will get practice at planning, preparing visual aids, and then presenting a topic of their choosing.

To help students complete the portfolio described in Assessing What You've Learned, suggest that they consider for evaluation their work on Lesson 3.1, Exercise 6; Lesson 3.2, Exercise 6; Lesson 3.3, Exercise 10; Lesson 3.4, Exercise 6; Lesson 3.5, Exercise 10; and Lesson 3.6, Exercise 9.